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THE
DOCTRINE
AND
APPLICATION
OF
FLUXIONS.

CONTAINING

(Besides what is common on the Subject)

A Number of NEW IMPROVEMENTS in the THEORY.

AND

The SOLUTION of a Variety of New, and very Interesting, Problems in different Branches of the MATHEMATICKS.

PART I.

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TO THE

RIGHT HONOURABLE

George Earl of *Macclesfield.*

MY LORD,

AS I esteem it a very great Honour to be permitted to place the following Sheets under your Lordship's Protection, who are not only an Encourager of, but an Ornament to, Mathematical Learning; I have taken more than ordinary Pains, that, *What* is here usher'd into the World, with such Advantage, may not be found altogether unworthy of so distinguished a Patron.

I am not vain enough to imagine, that, to One so deeply read in *these* abstruse and curious Speculations, as your Lordship is universally allow'd to be, this Work will appear without Faults: But then, I have the Satisfaction to think, on the other hand, that, whatever is Here to be met with capable of bearing the Test of an exact and solid Judgment, will *also* have its due Weight, and not fail of receiving your Lordship's Approbation: And if, upon the Whole, there is Merit enough found to intitle me to a favourable Reception, it will gratify the highest Ambition of,

MY LORD,

Your LORDSHIP'S

Most Obedient Humble Servant,

Tho. Simpson.

PREFACE.

HAVING, in the Year 1737, publiſhed a Piece, on this ſame Subject, under the Title of *A Treatiſe of Fluxions* (whereof the whole Impreſſion hath been long ſince ſold) it may be proper here, firſt of all, to aſſign the Reaſons why this Work is ſent abroad into the World as a New Book, rather than a Second Edition of the ſaid Treatiſe. Which, in ſhort, are theſe two: Firſt, becauſe the preſent Work is vaſtly more full and comprehenſive; and, ſecondly, becauſe the principal Matters in it which are alſo to be met with in that Treatiſe, are handled in a different Manner.

BESIDES the Preſs-Errors with which the ſaid Treatiſe abounds, there are ſeveral Obſcurities and Defects (which the Author's Want of Experience, and the many Diſadvantages he then labour'd under, in his firſt Sally, may, it is hoped, in ſome meaſure excuſe.) But what is now offer'd to the Publick, being a Performance of more mature Consideration and Judgment, it will, I flatter myself, be found much more correct, and claim a favourable Reception; especially, as particular Care and Pains have been taken to put every Thing in a clear Light, and to oblige the lower, as well as the more experienc'd, Class of Readers.

THE Notion and Explication *Here* given of the first Principles of Fluxions, are not essentially different from what they are in the above-mention'd Treatise, tho' expressed in other Terms. The Consideration of Time, which I have introduced into the General Definition, will, perhaps, be disliked by *Those* who would have Fluxions to be *meer Velocities*: But the Advantage of considering them *otherwise* (not as the Velocities Themselves, but the Magnitudes *They* would, uniformly, generate in a given finite Time) appear to me sufficient to obviate any Objection on that Head.

BY taking Fluxions as *meer Velocities*, the Imagination is confin'd, as it were, to a Point, and, without proper Care, insensibly involv'd in metaphysical Difficulties: But according to our Method of conceiving and explaining the Matter, less Caution in the Learner is necessary, and the higher Orders of Fluxions are render'd much more easy and intelligible——Besides, tho' Sir

Isaac

Isaac Newton defines Fluxions to be *the Velocities of Motions*, yet He hath Recourse to the Increments, or Moments, generated in equal Particles of Time, in order to determine those Velocities; which he afterwards teaches us to expound by finite Magnitudes of other Kinds: Without which (as is already hinted above) we could have but very obscure Ideas of the higher Orders of Fluxions: For if Motion in (or at) a Point be so difficult to conceive, that, *Some* have, even, gone so far as to dispute the very Existence of Motion, how much more perplexing must it be to form a Conception, not only, of the Velocity of a Motion, but also infinite Changes and Affections of *It*, in one and the same Point, where all the Orders of Fluxions are to be considered.

SEEING the Notion of a Fluxion, according to our Manner of defining It, supposes an uniform Motion, it may, perhaps, seem a Matter of Difficulty, at first View, how the Fluxions of Quantities, generated by Means of accelerated and retarded Motions, can be rightly assigned; since not any, the least, Time can be taken during which the generating Celerity continues the same: Here, indeed, we cannot express the Fluxion by any Increment or Space, *actually*, generated in a given Time (as in uniform Motions.) But, then, we can easily determine, what the contemporary Increment, or generated Space *would be*, if the Acceleration, or Retardation, was to cease

at

at the propoſed Poſition in which the Fluxion is to be found: Whence the true Fluxion, itſelf, will be obtained; without the Aſſiſtance of infinitely ſmall Quantities, or any metaphyſical Conſiderations.

THUS, for Example, the Motion of a Ball, deſcending by the Force of its own Gravity, is continually accelerated; but to have the Fluxion of the Diſtance fall'n thro' at any given Poſition of the Ball, we muſt find how far the Ball *would*, uniformly, deſcend, from that Point, in a given Time, if the Gravity, or the Earth's Attraction, from thence, was to ceaſe acting. By which Means we ſhall have as clear an Idea of the Fluxion and the true Meaſure of the Velocity of the Ball, at any Point aſſigned, as in thoſe Caſes where the Motion is, *actually*, uniform.

AGAIN, if a Right-line be ſuppoſed to move parallel to itſelf with an equable Motion, and to increaſe in Length, at the ſame Time; the Area generated thereby, will increaſe with an accelerated Velocity: But the Fluxion thereof, at any given Poſition of the Line, will be had by taking that Part of the Increment which *would*, uniformly, ariſe, was the Length (as well as the Velocity) of the Line to continue invariable from the propoſed Poſition. For, if the Length be ſuppoſed to increaſe, from the ſaid Poſition, the Area generated, from thence, will be, evidently, greater than That which would uniformly ariſe in the ſame Time; ſince the new Parts, produced

each

each succeeding Moment, are greater and greater, Therefore the Fluxion must be less than any Space that can be described, in the given Time, when the Line increases. And, in the same Manner, the Fluxion will appear to be greater than any Space that can be described, in the same Time, when the Line decreases. *It* must, therefore, be equal to that Space, which will arise, when the Length of the generating Line, from the given Position, is supposed neither to increase nor decrease: Agreeable to *Art.* 4.

THUS much it seem'd proper to offer Here with regard to the First Principles—I shall now proceed to say something concerning the Order observ'd in treating, and putting together, the several Parts of the Work; wherein the Ease and Benefit of the young Beginner have been particularly consulted: To load such an One with a Multitude of Rules and Precepts, before giving him any Taste of their Use and Application, would, certainly, be very discouraging; and like obliging a Traveller to ascend an high Mountain, without allowing him to stop by the Way, to take Breath, and refresh his Spirits with a Prospect of the agreeable and extensive View he has to expect when he arrives at the Summit: I have therefore, after demonstrating the First Principles, proceeded immediately to exemplify their Utility in several entertaining Enquiries, before touching at all upon the Inverse Method, or the more dif-

ficult

ficult Parts of the Direct. And, since that Branch of the Inverse Method which treats of the Comparison of Fluents is, naturally, somewhat difficult, it is referred to the Second Part of the Work, together with such other Matters in the Theory as might appear, either, too tedious or hard to a Learner at first setting out. The like Care has been taken in the Disposal of the rest of the Work——As to the several Particulars whereof *It* is composed, I must refer to the Book itself, They being too many to be here enumerated: One Thing, however, I must not omit to take notice of, relating to that Part which treats of the aforesaid Business of Fluents: To which it may, perhaps, be objected, That, notwithstanding my having insisted so largely on the Subject, there are a Number of Forms of Fluxions and Fluents to be met with in Authors, that I have not so much as touch'd upon. This is granted; but then they are most of them such as, I dare pronounce, can never arise in any Inquiry into Nature: And it would, doubtless, be Time and Labour misapply'd, to swell the Work, and embarrass the Learner with a Number of unnecessary Difficulties, and empty Speculations; when what is, really, proper and useful, in the Subject, is sufficient (it is well known) to exercise his utmost Attention and Resolution.

I Cannot put an End to this Preface without acknowledging my Obligations to a small Tract, intitled, *An Explanation of Fluxions in a Short Essay on the Theory*; printed for *W. Innys*: Wrote by a worthy Friend of mine (who was too modest to put his Name to that, his first, Attempt) whose Manner of determining the Fluxion of a Rectangle, and illustrating the higher Orders of Fluxions, I have, in particular, follow'd, with little or no Variation.

THE

ERRATA.

Page 10. l. 15. for x^2y^2z read $x^2y^2\dot{z}$; p. 34. l. 1. r. *uniformly*; p. 49. l. 10. let the Comma before the Word *which* be put after it; p. 71. l. 13. for *Involute*, r. *Evolute*; p. 95. l. 21. for $\times p$, r. $+p$; p. 104. l. 3. for 0.01, r. 0.01111 *&c.* p. 109. l. 3. for a^2, r. $2a$; p. 128. l. last but one, r. $\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$; p. 137. l. 14. for AN, r. *AN drawn into AB*; p. 148. l. 14. for z and $\dot{z}$, r. x and $\dot{x}$; p. 150. l. 3. before $\frac{a\dot{z}}{\sqrt{a^2+z^2}}$ r. $=$; p. 153. l. 4. for x, r. $\dot{x}$; p. 157. the Letter m, in the Cut should stand lower, at the Intersection of the Curve; p. 160. l. 7. for a^{n-2}, r. a^{2n-2}; p. 172. l. 21. for py^2x, r. $py^2\dot{x}$; p. 215. l. 4. and 6. for OC^3, r. $OC^2 \times OG$; p. 253. l. 5. and 6. for CQ, r. Cq; p. 24. instead of l. 27. read, *which Equation being no longer possible than till*, &c.

THE DOCTRINE and APPLICATION OF FLUXIONS.

PART the Firft.

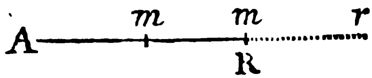
SECTION I.

Of the Nature, and Inveſtigation, of Fluxions.

1. IN Order to form a proper Idea of the Nature of Fluxions, all Kinds of Magnitudes are to be confidered as generated by the *continual* Motion of fome of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface.

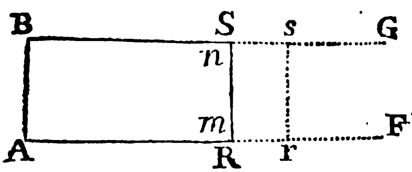
2. Every Quantity fo generated is called a variable, or flowing Quantity: *And the Magnitude by which any flowing Quantity* WOULD BE *uniformly increafed in a given Portion of Time, with the generating Celerity at any propofed Pofition, or Inftant (was it from thence to continue invariable) is the Fluxion of the faid Quantity at that Pofition, or Inftant.*

Thus, let the Point *m* be conceived to move from A, and generate the variable Right-line A*m*, by a Motion any how regulated; and



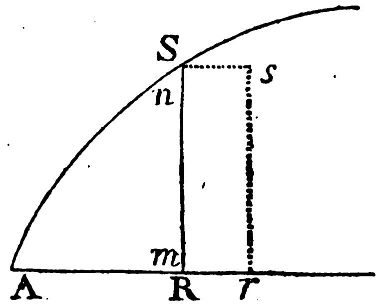
let the Celerity thereof, when it arrives at any proposed Position R, be such *as would*, was it to continue uniform from that Point, be sufficient to describe the Distance, or Line R*r*, in the given Time allotted for the Fluxion: Then will R*r* be the Fluxion of the variable Line A*m*, in that Position.

3. The Fluxion of a plane Surface is conceived in like Manner, by supposing a given Right-line *mn* to move parallel to itself, in the Plane of the parallel,



and immoveable Lines AF and BG: For, if (as above) Rr be taken to express the Fluxion of the Line A*m*, and the Rectangle RrsS be completed; then that Rectangle, being the Space which *would be* uniformly described by the generating Line *mn*, in the Time that A*m* *would be* uniformly increased by *m*r, is therefore the Fluxion of the generated Rectangle B*m*, in that Position, according to the true Meaning of the Definition.

4. If the Length of the generating Line *mn* continually varies, the Fluxion of the Area will *still* be expounded by a Rectangle under that Line, and the Fluxion of the Abscissa, or Base: For let the curvilineal Space A*mn* be generated by the continual, and parallel, Motion of the (now) variable Line *mn*, and let Rr be the Fluxion of the Base, or Abscissa, *Am* (as before); then the Rectangle RrsS will, here also, be the Fluxion of the generated Space A*mn*: Because, if the Length and Velocity of the generating Line *mn* were to



to continue invariable from the Position RS, the Rectangle RrsS would then be uniformly generated, with the very Celerity wherewith it begins to be generated, or with which the Space A*mn* is increased in that Position..

5. From what has been hitherto said it will appear, that *the Fluxions of Quantities are, always, as the Celerities by which the Quantities themselves increase in Magnitude:* Whence it will not be difficult to form a Notion of the Fluxions of Quantities otherwise generated; as well such as arise from the Revolution of Right-lines and Planes, as those by parallel Motion: But of this hereafter. I come now to shew the Manner of determining the Fluxions of algebraic Quantities; by which all others, of what Kind soever, are explicable. But first of all it will be requisite to premise the following Observations.

I. *That the final Letters* u, w, x, y, z *of the Alphabet are commonly put for variable Quantities; and the initial Letters* a, b, c, d, &c. *for invariable ones:* Thus the Diameter of a given Circle may be denoted by a, and the Sine of any Arch thereof (considered as variable) by x.

II. *That the Fluxion of a Quantity represented by a single Letter, is usually expressed by the same Letter with a Dot or Full-point over it:* Thus the Fluxion of x is represented by $\dot{x}$, and that of y by $\dot{y}$.

III. *That the Fluxion of a Quantity which decreases, instead of increasing, is to be considered as negative.*

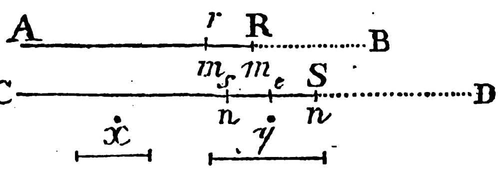
PROPOSITION I.

6. *The Fluxion of a Quantity being given, 'tis proposed to find the Fluxion of any Power of that Quantity.*

As a clear underſtanding of this Problem will be of great Importance throughout the whole Work, it may not be improper to conſider it firſt in one or two of its moſt ſimple Caſes.

Caſe 1. Let $\dot{x}$ expreſs the Fluxion of x, (according to the foregoing Notation) and let the Fluxion of x^2 be required.

Conceive two Points m and n to proceed, at the ſame time, from two other Points A and C, along the Right-lines AB and CD, in ſuch ſort, that the Meaſure of the Diſtance CS (y), deſcribed by the latter, may be, *always*, equal to the Square of that AR (x), deſcribed by the former moving uniformly.



Furthermore, let r, s, and R, S, be any contemporary Poſitions of the generating Points, and let the Lines $\dot{x}$ and $\dot{y}$ repreſent the reſpective Diſtances that *would be* uniformly deſcribed, in the ſame time, with the Celerities of thoſe Points at R and S, then thoſe Lines will expreſs the Fluxions of Am and Cn in this Poſition, (*by the Definition*, Art. 2 and 5).

Moreover, ſince $Cs = Ar^2$ and $CS = AR^2$ (*by Hypotheſis*), if Rr be denoted by v, we ſhall have CS $(y) = x^2$, and Cs $(= \overline{x-v}^2) = x^2 - 2xv + v^2$, and conſequently S$s$ $(= CS - Cs) = 2xv - v^2$; from whence we gather, that, while the Point m moves over the Diſtance v, the Point n moves over the Diſtance

$2xv$

$2xv - v^2$. But this last Distance (since the Square of any Quantity is known to increase faster, in Proportion, than the Root) is not described with an uniform Motion (like the former), but an accelerated one; and therefore is equal to, and may be taken to express, the uniform Space that might be described with the mean Celerity at some intermediate Point *e*, in the same time. Therefore, seeing the Distances that might be described, in equal times, with the uniform Celerity of *m*, and the mean Celerity at *e*, are to each other as v to $2xv - v^2$, or as 1 to $2x - v$, or, lastly, as $\dot{x}$ to $2x\dot{x} - v\dot{x}$, (all which are in the same Proportion) it is evident, that, in the time the Point *m* would move uniformly over the Distance $\dot{x}$, the other Point *n*, with its Celerity at *e*, *would* move uniformly over the Distance $2x\dot{x} - v\dot{x}$. This being the Case, let *r*, R, and *s*, S, be now supposed to coincide, by the Arrival of the generating Points at R and S, then *e* (being always between *s* and S) will likewise coincide with S; and the Distance, $2x\dot{x} - v\dot{x}$, which might be uniformly described in the aforesaid time, with the Velocity at *e*, (now at S), will become barely equal to $2x\dot{x}$; which *(by the Defin.)* is equal to ($\dot{y}$), the true Fluxion of C*n* or x^2 [a].

[a] *It may, perhaps, seem inaccurate, that the Fluxions of x and x² are compared together, and expressed both by Lines, when the flowing Quantities themselves, considered as a Right Line and a Square, admit of no Comparison.——This Objection would, indeed, be of force, were the Expressions restrained to a geometrical Signification; but here our Notions are more abstracted and universal, not obliging us to regard what Kind of Extension, may be defined by this or that Expression, but only the Values of the algebraic Quantities thereby signified; to which the Measures of all other Quantities whatever are ultimately referred.——And, though Quantities of different Kinds cannot be compared with each other, their Measures, in Numbers, may.——Thus, for Instance, though it would be wrong to affirm, that a Square whose Area is 9 Inches is equal to a Line of 9 Inches long, yet it is no Impropriety at all to say the Numbers expressing their Measures, in Inches, are equal.*

7. *Caſe* 2. Let the Fluxion of x^3 be required.

Suppoſe every Thing to remain as in the preceding Caſe; only let Cn be here equal to the Cube of Am (inſtead of the Square).

Then, in the very ſame manner, we have Ss ($=$CS $-$C$s = x^3 - \overline{x-v}|^3) = 3x^2v - 3xv^2 + v^3$: From whence it appears, that the Diſtances which *might be* deſcribed, in the ſame time, with the uniform Celerity of m, and the mean Celerity at e, will, in this Caſe, be to each other as v to $3x^2v - 3xv^2 + v^3$, or as $\dot{x}$ to $3x^2\dot{x} - 3xv\dot{x} + v^2\dot{x}$: Which laſt Expreſſion, when s, e, and S coincide (as before) will become $3x^2\dot{x}$, the true Fluxion of x^3 required.

8. *Univerſally*. Let Cn be, *always*, equal to $\overline{Am}|^n$; alſo let $\overline{x-v}|^n$ (or $x-v$ raiſed to the Power whoſe Exponent is n) be repreſented by $x^n - ax^{n-1}v + bx^{n-2}v^2 - cx^{n-3}v^3$, &c. and let every Thing elſe be ſuppoſed as above.

Then, ſince Ss $(x^n - \overline{x-v}|^n)$ is $= ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c. it is plain that the Spaces which might be deſcribed, in the ſame time, with the uniform Celerity of m, and the mean Celerity at e, will, here, be to each other as v to $ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c, or as $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x} + cx^{n-3}v^2\dot{x}$, &c.

Therefore, all the Terms, wherein v is found, vaniſhing, when s, e, and S coincide, we have $ax^{n-1}\dot{x}$ for the required Fluxion of Cn, or x^n; which Fluxion, becauſe the numeral Co-efficient of the ſecond Term of a Binomial involved is known to be, *univerſally*, equal to the Exponent of the Power, will alſo be truly expreſſed by $nx^{n-1}\dot{x}$. Q. E. I.

9. If the Quantity Am (or x) be generated with an accelerated, or a retarded Motion, inſtead of an uniform

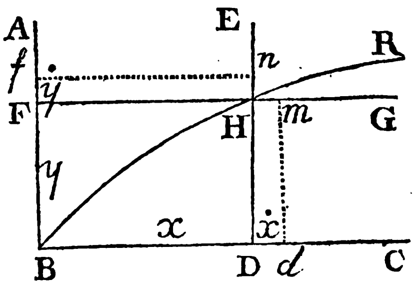
form one, the Fluxion of x^n (or C*n*) will come out exactly the same:

For the Spaces *r*R and *s*S, actually described in the same time, being always, to each other, in the Ratio of $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}\dot{v}x$, &c. the mean Celerities, at certain intermediate Points between *r*, R and *s*, S must, also, be in that Ratio: Which, when *v* vanishes (as above) will become that of $\dot{x}$ to $ax^{n-1}\dot{x}$, (or $nx^{n-1}\dot{x}$) the very same as before.

PROPOSITION II.

10. *To find the Fluxion of the Product or Rectangle of two variable Quantities.*

Conceive two Right-lines DE and FG, perpendicular to each other, to move, from two other Right-lines, BA and BC, continually parallel to themselves, and thereby generate the Rectangle DF. Let the Path of their



Interfection, or the Loci of the Angle H, be the Line BHR; also let D*d* ($\dot{x}$) and F*f* ($\dot{y}$) be the Fluxions of the Sides BD (*x*) and BF (*y*), and let *dm* and *fn*, parallel to DH and FH, be drawn. Therefore, because the Fluxion of the Space or Area BDH is truly expressed by the Rectangle D*m* ($= y\dot{x}$ *) and that of the Area, or Space BFH, by the Rectangle F*n*, and equal Quantities have equal Fluxions, it follows that the Fluxion of the Rectangle xy=DF (=BDH+BFH) is truly expressed by $y\dot{x}+x\dot{y}$. Q. E. I. * Art. 4.

The same otherwise.

11. Let xy be the given Rectangle (as before); and put $z = x + y$, then z^2 being $= x^2 + 2xy + y^2$, we have $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 = xy$. But the Fluxion of $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, (and consequently that of its Equal xy) is $z\dot{z} - x\dot{x} - y\dot{y}$ (by *Art.* 6): Which, because $z = x + y$ and $\dot{z} = \dot{x} + \dot{y}$, is also equal to $\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$. Q. E. I.

Corollary 1.

12. Hence the Fluxion of the Product of three variable Quantities (yzu) may be derived: For, if x be put $= zu$; then yzu will become $= yx$, and its Fluxion $= y\dot{x} + x\dot{y}$ (*as above*:) But x being $= zu$, and, therefore, $\dot{x} = \dot{z}u + u\dot{z}$, if these Values be substituted in $y\dot{x} + x\dot{y}$, it will become $y \times \overline{z\dot{u} + u\dot{z}} + zu\dot{y} = yz\dot{u} + yu\dot{z} + zu\dot{y}$ the Fluxion of yzu required. In like Manner the Fluxion of $xyzu$ will appear to be $\dot{x}yzu + x\dot{y}zu + xy\dot{z}u + xyz\dot{u}$, and that of $xyzuw = xyzu\dot{w} + xyz\dot{u}w + xy\dot{z}uw + x\dot{y}zuw + \dot{x}yzuw$.

Corollary 2.

13. Hence, also, the Fluxion of a Fraction $\frac{u}{z}$ may be determined. For, putting $x = \frac{u}{z}$, we have $xz = u$, and therefore $x\dot{z} + z\dot{x} = \dot{u}$ (*as above*); whence, by Transposition and Division, $\dot{x} = \frac{\dot{u}}{z} - \frac{x\dot{z}}{z} = \frac{\dot{u}}{z} - \frac{u\dot{z}}{z^2}$ (by writing $\frac{u}{z}$ for x) $= \frac{z\dot{u} - u\dot{z}}{z^2}$; which is the true Fluxion of x, or its Equal $\frac{u}{z}$, the Fraction proposed.

14. Now, from the foregoing Propositions, and their subsequent Corollaries, the following practical Rules, for

for determining the Fluxions of algebraic Quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a variable Quantity.

Multiply the Fluxion of the Root by the Exponent of the Power, and the Product by that Power of the same Root whose Exponent is less by Unity than the given Exponent.

This Rule is investigated in Prop. 1, and is nothing more than $n x^{n-1} \dot{x}$ (the Fluxion of x^n) expressed in Words.

Hence the Fluxion of x^3 is $3x^2\dot{x}$; that of x^5 is $5x^4\dot{x}$; and that of $\overline{a+y}|^7$ is $7\dot{y} \times \overline{a+y}|^6$, (because, a being constant, $\dot{y}$ is the true Fluxion of the Root $a+y$, in this Case).

Moreover the Fluxion of $\overline{a^2+z^2}|^{\frac{3}{2}}$, will be $\frac{3}{2} \times 2z\dot{z} \times \overline{a^2+z^2}|^{\frac{1}{2}}$, or $3z\dot{z}\sqrt{a^2+z^2}$: For here, x being put $= a^2+z^2$, we have $\dot{x} = 2z\dot{z}$, and therefore $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, the Fluxion of $x^{\frac{3}{2}}$ (or $\overline{a^2+z^2}|^{\frac{3}{2}}$) is $= 3z\dot{z}\sqrt{a^2+z^2}$, as above).

RULE II.

15. To find the Fluxion of the Product of several variable Quantities multiplied together.

Multiply the Fluxion of each, by the Product of the rest of the Quantities, and the Sum of the Products thus arising will be the Fluxion sought *.

* Art. 12.

Thus the Fluxion of xy, is $\dot{x}y + \dot{y}x$; that of xyz, is $\dot{x}yz + \dot{x}z\dot{y}$ $+ \dot{y}z\dot{x}$; and that of $xyzu$, is $\dot{x}yzu + \dot{x}yu\dot{z} + xzu\dot{y} + yzu\dot{x}$.

RULE

RULE III.

16. To find the Fluxion of a Fraction.

From the Fluxion of the Numerator drawn into the Denominator, subtract the Fluxion of the Denominator drawn into the Numerator, and divide the Remainder by * Art. 13. *the Square of the Denominator* *.

Thus, the Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{y^2}$; that of $\frac{x}{x+y}$, is

$$\frac{\dot{x}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times x}{\overline{x+y}|^2}=\frac{y\dot{x}-x\dot{y}}{\overline{x+y}|^2};$$ and that of $\frac{x+y+z}{x+y}$,

or $1+\frac{z}{x+y}$, is $\frac{\dot{z}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times z}{\overline{x+y}|^2}$; and so of others.

17. In the Examples hitherto given, each is resolved by its own particular Rule; but in those that follow, the Use of two, and sometimes of all the three, Rules is requisite.

Thus (by Rule 1. and 2.) the Fluxion of x^2y^2 is $2x^2y\dot{y}+2y^2x\dot{x}$; that of $\frac{x^2}{y^2}$ is $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$, (*by Rule* 1. *and* 3.) and that of $\frac{x^2y^2}{z}$ is $\frac{\overline{2x^2y\dot{y}+2y^2x\dot{x}}\times z-x^2y^2\dot{z}}{z^2}$; where all the three Rules are necessary.

When the proposed Quantity is affected by a Co-efficient, or constant Multiplicator, the Fluxion found as above, must be multiplied by that Co-efficient or Multiplicator.

Thus, the Fluxion of $5x^3$ is $15x^2\dot{x}$. For, the Fluxion of x^3 being $3x^2\dot{x}$, that of $5x^3$, which is 5 times as great, must consequently be $5\times3x^2\dot{x}$, or $15x^2\dot{x}$.

And, in the very same Manner the Fluxion of ax^n will appear to be $nax^{n-1}\dot{x}$. Moreover, the Fluxion of $\frac{a}{\overline{x^2+y^2}|^{\frac{1}{2}}}$, or $a\times\overline{x^2+y^2}|^{-\frac{1}{2}}$, will be expressed by

$$a\times -\frac{1}{2}\times \overline{2x\dot{x}+2y\dot{y}}\times\overline{x^2+y^2}^{\,-\frac{3}{2}}, \text{ or } -\frac{a\times \overline{x\dot{x}+y\dot{y}}}{\overline{x^2+y^2}^{\,\frac{3}{2}}};$$

that of $\sqrt{x+y^{\frac12}}$, or $\overline{x+y^{\frac12}}^{\,\frac12}$, by $\overline{\frac12\dot{x}+\frac12\times\frac12\dot{y}y^{-\frac12}}\times \overline{x+y^{\frac12}}^{\,-\frac12}$, (*Rule* 1.) or $\frac{\frac12\dot{x}+\frac14\dot{y}y^{-\frac12}}{\sqrt{x+y^{\frac12}}}$, or $\frac{\frac12\dot{x}y^{\frac12}+\frac14\dot{y}}{\sqrt{xy+y^{\frac32}}}$;

and that of $\frac{\overline{x+a}^{\,2}}{\sqrt{x^2-a^2}}$, or $\frac{\overline{x+a}^{\,2}}{\overline{x^2-a^2}^{\,\frac12}}$, by

$$\frac{2\dot{x}\times\overline{x+a}\times\overline{x^2-a^2}^{\,\frac12}-x\dot{x}\times\overline{x^2-a^2}^{\,-\frac12}\times\overline{x+a}^{\,2}}{x^2-a^2}; \text{ which}$$

by Reduction, is $=\frac{2\dot{x}\times\overline{x^2-a^2}^{\,\frac12}-x\dot{x}\times\overline{x^2-a^2}^{\,-\frac12}\times\overline{x+a}}{x-a}$

$$=\frac{2\dot{x}\times\overline{x^2-a^2}-x\dot{x}\times\overline{x+a}}{\overline{x-a}\times\overline{x^2-a^2}^{\,\frac12}}=\frac{2\dot{x}\times\overline{x-a}\times\overline{x+a}-x\dot{x}\times\overline{x+a}}{\overline{x-a}\times\sqrt{x^2-a^2}}$$

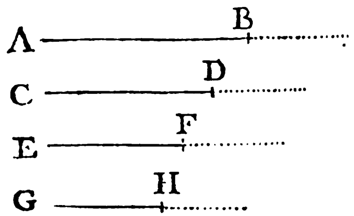
$$=\frac{\overline{x+a}\times\overline{x\dot{x}-2a\dot{x}}}{\overline{x-a}\times\sqrt{x^2-a^2}}.$$

Having explained the Manner of confidering and determining the firft Fluxions of variable or flowing Quantities, it will be proper to fay fomething, now, concerning the higher Orders, as Second, Third, Fourth, &c. Fluxions.

18. *The Second Fluxion of a Quantity is the Fluxion of the variable or algebraic Quantity expreffing the Firft Fluxion already defined *. By the Third Fluxion is meant the Fluxion of the variable Quantity expreffing the Second: And by the Fourth, the Fluxion of the variable Quantity expreffing the Third Fluxion: And fo on.*

* Art. 2.

Thus, for Example, let the Line AB reprefent a variable Quantity, generated by the Motion of the Point B, and let the (firft) Fluxion thereof (or the Space that *might be* uniformly defcribed in a given Time, with the Celerity of B) be always expreffed by the Diftance of of the Point D from a given, or fixed Point C: Then, if the Celerity of B be not every where the same; the Distance CD, expressing the Measure of that Celerity, must also vary, by the Motion of D, from, or towards C, according as the Celerity of B is an increasing or a decreasing one: And the Fluxion of the Line CD, so varying (or the Space (EF) that *might be* uniformly described in the aforesaid given Time, with the Celerity of D) is the second Fluxion of AB. Again, if the Motion of B be such that neither it, nor that of D, (which depends upon it) be equable, then EF, expressing the Celerity of D, will also have its Fluxion GH; which is the third Fluxion of AB, and the second Fluxion of CD.



And thus are the Fluxions of every other Order to be considered, *being the Measures of the Velocities by which their respective flowing Quantities, the Fluxions of the preceding Order, are generated* *.

* Art. 2.

19. Hence it appears, that a second Fluxion always shews the Rate of the Increase, or Decrease, of the first Fluxion; and that Third, Fourth, &c. Fluxions, differ in Nothing (except their Order and Notation) from First Fluxions, being actually such to the Quantities from whence they are immediately derived; and therefore are also determinable, in the very same Manner, by the general Rules already delivered.

Thus, *by Rule* 3. the (first) Fluxion of x^3 is $3x^2\dot{x}$: And, if $\dot{x}$ be supposed constant, that is, if the Root x be generated with an equable Celerity, the Fluxion of $3x^2\dot{x}$ (or $3\dot{x} \times x^2$) again taken, by the same Rule, will be $3\dot{x} \times 2x\dot{x}$, or $6x\dot{x}^2$; which therefore is the second Fluxion of x^3: Whose Fluxion, found in like Sort, will be $6\dot{x}^3$, the third Fluxion of x^3. Further than which

which we cannot go in this Cafe, becaufe the laft Fluxion $6\dot{x}^3$ is here a conftant Quantity.

20. In the preceding Example the Root x is fuppofed to be generated with an equable Celerity: But, if the Celerity be an increafing or a decreafing one, then $\dot{x}$, expreffing the Meafure thereof, being variable, will alfo have its Fluxion; which is ufually denoted by $\ddot{x}$: Whofe Fluxion, according to the fame Method of Notation, is again defigned by $\dddot{x}$; and fo on, with refpect to the higher Orders.

21. Here follow a few Examples, wherein the Root x, (or y) is fuppofed to be generated with a variable Celerity.

Thus, the firft Fluxion of x^3 is $3x^2\dot{x}$ (or $3x^2 \times \dot{x}$). And, if the Fluxion of $3x^2 \times \dot{x}$ (confidered as a Rectangle) be, again, found (*by Rule* 2.) we fhall have $6x\dot{x} \times \dot{x} + 3x^2 \times \ddot{x} = 6x\dot{x}^2 + 3x^2\ddot{x}$, for the fecond Fluxion of x^3.

Moreover, from the Fluxion laft found we fhall in like manner get $6\dot{x} \times \dot{x}^2 + 6x \times 2\dot{x}\ddot{x} + 6x\dot{x} \times \ddot{x} + 3x^2 \times \dddot{x}$ (or $6\dot{x}^3 + 18x\dot{x}\ddot{x} + 3x^2\dddot{x}$) for the third Fluxion of x^3.

Thus alfo, if $\dot{y} = nx^{n-1}\dot{x}$, then will $\ddot{y} = n \times \overline{n-1} \times x^{n-2}\dot{x}^2 + n\ddot{x}x^{n-1}$; and if $\dot{z}^2 = \dot{x}\dot{y}$, then will $2\dot{z}\ddot{z} = \dot{x}\ddot{y} + \dot{y}\ddot{x}$: And fo of others. But, in the Solution of Problems, it will be convenient to make the firft Fluxion of fome one of the fimple Quantities (x or y) invariable, not only to avoid Trouble, but that it may ferve as a Standard to which the variable Fluxions of the other Quantities, depending thereon, may be always referred. The Reader is alfo defired here (once for all) to take particular Notice, that *the Fluxions of all Kinds and Orders, whatever, are contemporaneous, or fuch as may be generated together, with their refpective Celerities, in one and the fame Time.*

SECT.

SECTION II.

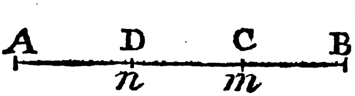
Of the Application of Fluxions to the Solution of Problems DE MAXIMIS ET MINIMIS.

22. IF a Quantity, conceived to be generated by Motion, increases, or decreases, 'till it arrives at a certain Magnitude or Position, and then, on the contrary, grows lesser or greater, and it be required to determine the said Magnitude or Position, the Question is called a Problem *de Maximis & Minimis*.

GENERAL ILLUSTRATION.

Let a Point *m* move uniformly in a Right Line, from A towards B, and let another Point *n* move after it, with a Velocity either increasing, or decreasing, but so that it may, at a certain Position, D, become equal to that of the former Point *m*, moving uniformly.

This being premised, let the Motion of *n* be first considered as an increasing one; in which Case the Distance of *n* behind *m* will continually



increase, 'till the two Points arrive at the cotemporary Positions C and D; but afterwards it will, again, decrease; for the Motion of *n*, 'till then, being slower than at D, it is also slower than that of the preceding Point *m* (by Hypothesis) but becoming quicker, afterwards, than that of *m*, the Distance *mn* (as has been already said) will again decrease: And therefore is a *Maximum*, or the greatest of all, when the Celerities of the two Points are equal to each other.

But, if *n* arrives at D with a decreasing Celerity; then its Motion being first swifter, and afterwards slower, than that of *m*, the Distance *mn* will first decrease and then

then increase; and therefore is a *Minimum*, or the least of all, in the forementioned Circumstance.

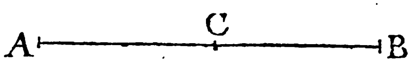
Since then the Distance *mn* is a *Maximum* or a *Minimum*, when the Velocities of *m* and *n* are equal, or when that Distance increases as fast through the Motion of *m*, as it decreases by that of *n*, its Fluxion at that Instant is evidently equal to Nothing *. Therefore, as the Motion of the Points *m* and *n* may be conceived such that their Distance *mn* may express the Measure of any variable Quantity whatever, it follows, that the Fluxion of any variable Quantity whatever, when a Maximum or Minimum, is equal to Nothing.

* Art. 2 and 5.

EXAMPLE I.

23. *To divide a given Right-line* AB *into two such Parts*, AC, BC, *that their Product, or Rectangle, may be the greatest possible.*

Put the given Line AB $= a$, and let the Part AC,



considered as variable (by the Motion of C from A towards B) be denoted by x: Then BC being $= a-x$, we have $AC\times BC = ax-x^2$: Whose Fluxion $a\dot{x}-2x\dot{x}$ being put $= 0$, according to the prescript, we get $a\dot{x} = 2x\dot{x}$, and consequently $x = \frac{1}{2}a$. Therefore AC and BC, in the required Circumstance, are equal to each other: Which we also know from other Principles.

EXAMPLE II.

24. *To find the Fraction which shall exceed its Cube by the greatest Quantity possible.*

Let x denote a variable Quantity, expressing Number in general; then the Excess of x above x^3 being universally represented by $x-x^3$, if the Fluxion thereof be taken, &c. we shall have $\dot{x}-3x^2\dot{x}=0$; and therefore $x=\sqrt{\frac{1}{3}}$, the Fraction required.

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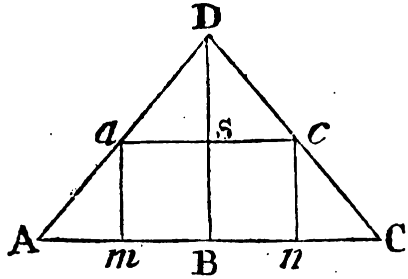
EXAMPLE III.

25. *To determine the greatest Rectangle that can be inscribed in a given Triangle.*



Put the Base AC of the given Triangle $= b$, and its Altitude BD $= a$; and let the Altitude (BS) of the inscribed Rectangle *mc* (considered as variable) be denoted by x: Then, because of the parallel Lines AC, and *ac*, it will be as BD (a) : AC (b) :: DS $(a-x)$: $\frac{ba-bx}{a} = ac$: Whence the Area of the Rectangle, or $ac \times BS$ will be $= \frac{bax-bx^2}{a}$: Whose Fluxion $\frac{ba\dot{x}-2bx\dot{x}}{a}$ being (as before) put $=0$, we shall get $x = \frac{1}{2}a$. Whence the greatest inscribed Rectangle is that whose Altitude is just half the Altitude of the Triangle.

26. It will be proper to observe *here*, that the Value of a Quantity, when a *Maximum* or *Minimum*, may oftentimes be determined with more Facility by taking the Fluxion of some given Part, Multiple, or Power, thereof, than from the Fluxion of the Quantity itself. Thus, in the preceding Example, where the general Expression is $\frac{bax-bx^2}{a} = \frac{b}{a} \times \overline{ax-x^2}$, if the constant Multiplicator $\frac{b}{a}$ be rejected, we shall have $ax-x^2$; whose Fluxion $a\dot{x}-2x\dot{x}$ being put $=0$, we get $x = \frac{1}{2}a$, *the very same as before*.

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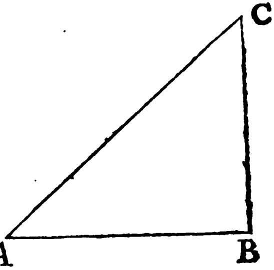
The Reason of which is obvious; because when the Quantity itself (be it of what Kind it will) is the greatest, or least possible, any *given* Part, Power, or Multiple of it is also the greatest or least possible.

EXAMPLE IV.

27. *Of all right angled plain Triangles having the same given Hypothenuse, to find that* (ABC) *whose Area is the greatest.*



Let AC$=a$, AB$=x$, and BC $=y$: Then, x^2+y^2 being $=a^2$, we shall have $y=\sqrt{a^2-x^2}$, and consequently $\frac{xy}{2}=\frac{x}{2}\sqrt{a^2-x^2}=$ the Area of the Triangle; whose Square $\frac{a^2x^2}{4}-\frac{x^4}{4}$ being, *also*, a *Maximum* *, the Fluxion thereof $\frac{a^2x\dot{x}}{2}-x^3\dot{x}$ must therefore be $=0$, † Whence x is found $=a\sqrt{\tfrac{1}{2}}$, and y $(\sqrt{a^2-x^2})=a\sqrt{\tfrac{1}{2}}$.

* Art. 26.

† Art. 22.

The same otherwise.

Since $\frac{1}{2}xy$ is a *Maximum*, and $x^2+y^2=a^2$, let the Fluxions of both be taken, and you will have $\frac{1}{2}x\dot{y}+\frac{1}{2}y\dot{x}=0$, and $2x\dot{x}+2y\dot{y}=0$; from the former of which $\dot{y}$ will be $=-\frac{y\dot{x}}{x}$; and from the latter, it will be $=-\frac{x\dot{x}}{y}$. Therefore $\frac{y\dot{x}}{x}$ and $\frac{x\dot{x}}{y}$ are equal to each other, and consequently $x=y$, (the same as before.)

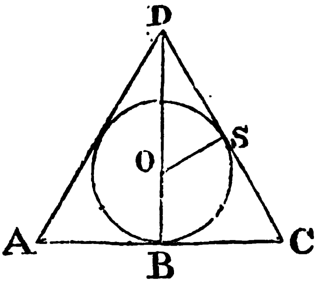
EXAMPLE V.

28. *Of all right angled plane Triangles containing* 1 *same given Area, to find that whereof the Sum of* 1 *two Legs* AB+BC *is the least possible.* (See the pr ceding Figure.)

Let one Leg, AB, be denoted by x, and the Ar of the Triangle by a; then the other Leg will be d noted by $\frac{2a}{x}$, and the Sum of the two Legs will be x $\frac{2a}{x}$; whereof the Fluxion is $\dot{x}-\frac{2a\dot{x}}{x^2}$; which, put = gives x (AB) $=\sqrt{2a}$: Whence BC $\left(\frac{2a}{x}\right)$ is also $\sqrt{2a}$. Therefore the two Legs are equal to ea other.

EXAMPLE VI.

29. *To determine the Dimensions of the least Isosceles T angle* ACD *that can circumscribe a given Circle.*



Let the Distar (OD) of the Vert of the Triangle fr the Center of the C cle, be denoted by and let the remaini Part of the Perpen cular, which is t Radius of the Circ be represented by Then, if OS, perp dicular to DC, be drawn, we shall have DS$=\sqrt{x^2-}$ and therefore, since DS : OS :: DB : BC, we likew have BC$=\frac{a\times\overline{x+a}}{\sqrt{x^2-a^2}}$; which multiplied by $\overline{x+a}$ (B

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gives $\frac{a\times\overline{x+a}|^2}{\sqrt{x^2-a^2}}$ for the Area of the Triangle: Which being a *Minimum*, its Square must be a *Minimum*, and consequently $\frac{\overline{x+a}|^4}{x^2-a^2}$, or its Equal $\frac{\overline{x+a}|^3}{x-a}$, a *Minimum* also *. Whose Fluxion, therefore, which is $\frac{3\dot{x}\times\overline{x+a}|^2\times\overline{x-a}-\dot{x}\times\overline{x+a}|^3}{\overline{x-a}|^2}$, being put $=0$, and the Whole divided by $\frac{\dot{x}\times\overline{x+a}|^2}{\overline{x-a}|^2}$, we also get $3\times\overline{x-a}-\overline{x+a}=0$; whence $x=2a$: Therefore, OD being $=2$OS, and the Triangles ODS and BDC equiangular, it is evident that DC is likewise $=2$BC$=$AC; and so the Triangle ACD, when the least possible, is equilateral.

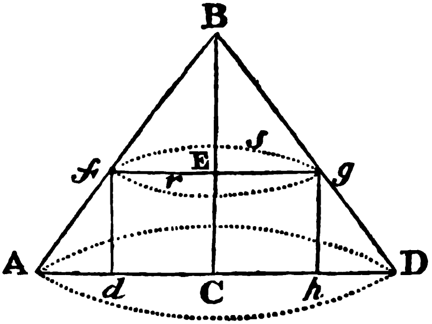
* Art. 26.

EXAMPLE VII.

30. *To determine the greatest Cylinder, dg, that can be inscribed in a given Cone* ADB.

Let $a=$BC, the Altitude of the Cone;
$b=$AD, the Diameter of its Base;
$x=fg$ (dh) the Diameter of the Cylinder, considered as variable;
$p=\left(\frac{3,14159, \&c.}{4}\right)$ the Area of the Circle whose Diameter is Unity.

Then, the Areas of Circles being to one another as the Squares of their Diameters, we have, $1^2 : x^2 :: p : (px^2)$ the Area of the Circle *fsgr*: Moreover, from the Similarity of the Triangles ABC and A*df*, we have $\frac{1}{2}b$ (AC) : a (BC) :: $\frac{1}{2}b-\frac{1}{2}x$ (Ad) : $df=\frac{ab-ax}{b}$; which multiplied by the Area px^2 (found above) gives

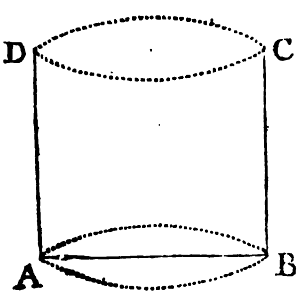


$\frac{pabx^2 - pa}{b}$ for the fol Content the Cyli der: Whi being *Maximun* its Fluxi $\frac{2pabx\dot{x}}{b}$ $\frac{3pax^2\dot{x}}{b}$ mu

* Art. 22. be $= 0$ *, consequently $x = \frac{2b}{3}$ and $df = \frac{a}{3}$: Fro whence it appears, that the inscribed Cylinder will b the greatest possible, when the Altitude thereof is ju $\frac{1}{3}$ of the Altitude of the whole Cone.

EXAMPLE VIII.

31. *To determine the Dimensions of a cylindric Measu* ABCD, *open at the Top, which shall contain a giv Quantity (of Liquor, Grain,* &c.) *under the least i ternal Superficies possible.*

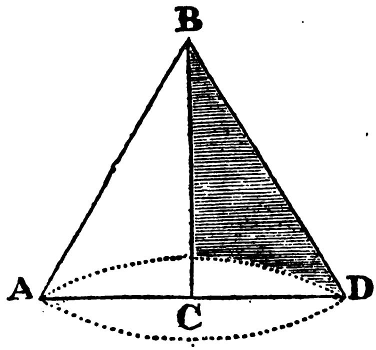


Let the Diamet AB$=x$, and the Alt tude AD$=y$; moreov let p (3,14159, &c denote the Periphery the Circle whose Di meter is Unity, and l c be the given Conte of the Cylinder. The it will be $1 : p :: x : (p$ the Circumference of t Base; which, multipli b by the Altitude y, gives pxy for the concave Superficies of the Cylinder. In like Manner, the Area of the Base, by multiplying the same Expression into $\frac{1}{4}$ of the Diameter x, will be found $=\frac{px^2}{4}$; which drawn into the Altitude y, gives $\frac{px^2y}{4}$ for the solid Content of the Cylinder; which being made $=c$, the concave Surface pxy will be found $=\frac{4c}{x}$, and consequently the whole Surface $=\frac{4c}{x}+\frac{px^2}{4}$: Whereof the Fluxion, which is, $-\frac{4c\dot{x}}{x^2}+\frac{px\dot{x}}{2}$, being put $=0$, we shall get $-8c+px^3=0$; and therefore $x=2\sqrt[3]{\frac{c}{p}}$: Further, because $px^3=8c$, and $px^2y=4c$, it follows, that $x=2y$; whence y is also known, and from which it appears, that the Diameter of the Base must be just the Double of the Altitude.

EXAMPLE IX.

32. *Of all Cones under the same given Superficies* (s) *to find that* (ABD) *whose Solidity is the greatest.*

Let the Semidiameter of the Base, AC$=x$, and the Length of the slant Side AB$=y$; and let p (as in the preceding Examples) denote the Periphery of the Circle whose Diameter is Unity.



 Then

Then the Circumference of the Base will be $=2px$, the Area of the Base $=px^2$, and the convex Superficies of the Cone $=pxy$, (which last is found by multiplying half the Periphery of the Base by the Length of the slant Side): Wherefore, since the whole Superficies is $=px^2+pxy=s$, we have $y=\frac{s}{px}-x$; whence the Altitude CB $(\sqrt{AB^2-AC^2})=\sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$; which multiplied by $\left(\frac{px^2}{3}\right)$ $\frac{1}{3}$ of the Area of the Base, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$ for the solid Content of the Cone. Which being a *Maximum*, its Square $\frac{s^2x^2}{9}-\frac{2psx^4}{9}$ must also be a *Maximum*; and therefore $\frac{2s^2x\dot{x}}{9}-\frac{8psx^3\dot{x}}{9}=0$; whence $s-4px^2=0$, and consequently $x=\sqrt{\frac{s}{4p}}$: From which $y\left(=\frac{s}{px}-x=\frac{s-px^2}{px}=\frac{3px^2}{px}=3x\right)$ will likewise be known; and from whence it will appear that the greatest Cone under a given Surface, (or a given Cone under the least Surface) will be when the Length of the slant Side is to the Semi-diameter of the Base in the Ratio of 3 to 1, or, (which comes to the same) when the Square of the Altitude is to the Square of the whole Diameter in the Ratio of 2 to 1.

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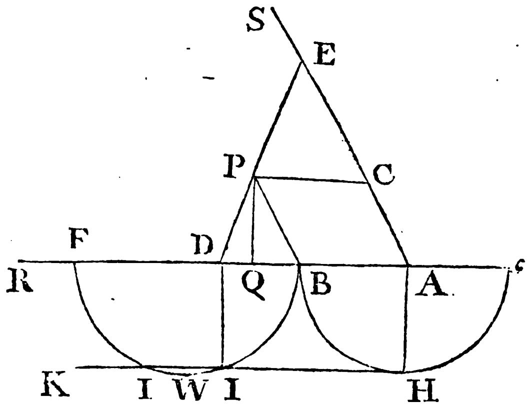
EXAMPLE X.

33. *To determine the Position of a Right-line* DE, *which, passing through a given Point* P, *shall cut two Right-lines* AR *and* AS, *given by Position, in such sort that the Sum of the Segments,* AD *and* AE, *made thereby, may be the least possible.*



Make PB, parallel to AS, $=a$, and PC, parallel to AR, $=b$; and let BD$=x$: Then, by reason of the parallel Lines, it will be, $x : a :: b : CE = \frac{ab}{x}$: Therefore AD+AE$=b+x+a+\frac{ab}{x}$, and its Fluxion, $\dot{x}-\frac{ab\dot{x}}{x^2}$, which, in the required Circumstance, being $=0$, we have x^2-ab also $=0$, and consequently $x=\sqrt{ab}$; whence the Position of DE is known. But the same Thing may be otherwise determined, independent of Fluxions, from the general Solution of the Problem for finding the Position of DE, when the Sum of the Segments AD and AE (instead of being a *Minimum*) shall be equal to a given Quantity. Of which Problem, the geometrical Construction may be as follows.

Compleat the Parallelogram ABPC (as before) and, in RA produced, take $Ac=AC$, and let cF be equal to the given Sum of the two Segments: Alſo let two Semi circles be deſcribed upon Bc and BF, and let AH, perpendicular to Bc, interſect the former in H; likewiſe let HK, parallel to Fc, interſect the latter in I; draw ID perpendicular to Fc, and, through P and D draw DE; which will be the Poſition required. For $AB\times Ac$ being $=AH^2=DI^2=BD\times DF$, we have BD : AB :: Ac (AC) : DF; alſo, becauſe of the parallel Lines, we have BD : AB :: AC : CE; whence DF= CE, and conſequently AD+AE (AD+AC+FD) is equal to cF, which Conſtruction is more neat than that in *p.* 155. of my *Geometry.* But to ſhew how far this may conduce to the Matter firſt propoſed; we are to obſerve, that, as the Problem here conſtructed appears to be impoſſible, when the Right line HK (inſtead of cutting or touching) falls wholly below the Circle BWF, the leaſt poſſible Value of BF (and conſequently of AD +AE) muſt, therefore, be when that Right-line touches the Circle; that is, when $BD=DI=AH=\sqrt{AB\times AC}$; which Value is the very ſame with that found above.

The ſame Concluſion may alſo be deduced from the algebraic Solution of the foreſaid Problem: For, putting $b+x+a+\frac{ab}{x}$ (AD+AE) $=s$, and ſolving the Equation, x will be found $=\frac{s-a-b}{2}\pm\sqrt{\frac{\overline{s-a-b}|^2}{4}-ab}$. Which Equation becoming impoſſible when $\frac{\overline{s-a-b}|^2}{4}$ $-ab$ is $=0$, we have x, in that Circumſtance, $=\frac{s-a-b}{2}=\sqrt{ab}$; *ſtill as before.* In like Manner the *Maxima* and *Minima* may be determined in other Caſes, by finding the Poſition or Circumſtance wherein the general Problem begins to be impoſſible, (ſuppoſing the Quantity ſought to be given). But the Operation by Fluxions

Fluxions is, for the general Part, much more short and expeditious.

EXAMPLE XI.

34. *The same being given as in the preceding Example, to determine the Position, when the Line* DE, *itself, is the least possible.*

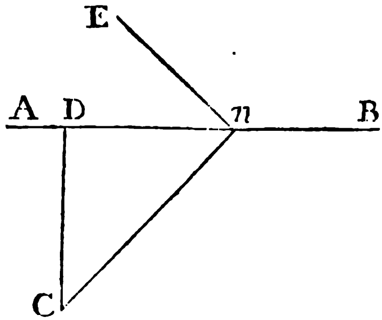
Upon AF let fall the perpendicular PQ; make BQ $=c$, and, the rest, as before: Then DP^2 being ($= DB^2+BP^2-2BQ\times DB$) $= x^2+a^2-2cx$, and $DB^2 : DP^2 :: DA^2 : DE^2$, we have $x^2 : x^2+a^2-2cx :: \overline{b+x}^2$ $: DE^2 = \frac{\overline{b+x}^2\times\overline{x^2-2cx+a^2}}{x^2} = \overline{b+x}^2\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}$; whose Fluxion, which is $2\dot{x}\times\overline{b+x}\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}+\overline{b+x}^2\times\overline{\frac{2c\dot{x}}{x^2}-\frac{2a^2\dot{x}}{x^3}}$, being put $=0$, and the whole Equation divided by $2\dot{x}\times\overline{b+x}$, there will come out $1-\frac{2c}{x}+\frac{a^2}{x^2}+\overline{b+x}\times\overline{\frac{c}{x^2}-\frac{a^2}{x^3}}=0$; whence $x^3-2cx^2+a^2x+\overline{b+x}\times\overline{cx-a^2}=0$; that is, (by Reduction) $x^3-cx^2+bcx-a^2b=0$: From the Resolution of which Equation, the Position of DE is determined.

LEMMA.

35. *If a Body or Point* (*n*) *be supposed to move in a Right-line* AB, *its absolute Celerity, in the Direction of that Line, will be to the relative Celerity, whereby it tends to, or from, a given Point* C, *any where out of the Line, as the Distance* C*n*, *is to the Distance* D*n*, *intercepted by n and the Perpendicular* CD; *or, as Radius to the Co-sine of the Angle of Inclination* D*n*C.

For, putting $CD=a$, $Dn=x$, and $Cn=y$, we have $a^2+x^2=y^2$, and consequently $2x\dot{x}=2y\dot{y}$ *: Whence

* Art. 2 and 5.



Whence $\dot{x}$: $\dot{y}$ :: y (Cn) : x (Dn) :: Radius : Co sine DnC : But, the Fluxions of Quantities are as the Celerities of their Increase *, therefore the Truth of the Proposition is manifest.

* Art. 2 and 5.

Corollary.

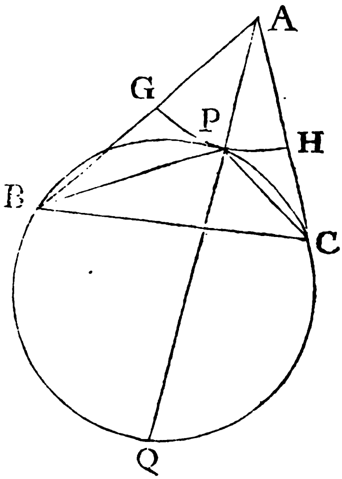
It follows from hence, that the relative Celerities in any two different Directions nE and nC, are directly as the Co-sines of the corresponding Angles DnE and DnC. Therefore, when nE is perpendicular to Cn, (and the Angle DnE therefore equal to C) the Celerity in the Direction nE, will be to that in the Direction nC, as the Sine of DnC is to its Co sine. From whence it appears, that the Celerities in the Directions Dn, Cn, and En (perpendicular to nC) are to each other as Cn, Dn, and CD respectively.

EXAMPLE XII.

36. *To determine the Position of a Point, from whence, if three Right-lines be drawn to so many given Points* A, B, C, *their Sum shall be the least possible.*

Let HPG be the Periphery of a Circle described about the Point A, as a Center, at any Distance AG; in which let the Point P be conceived to move with an uniform Celerity, from G towards H. Then, because the relative Celerity thereof, in the Direction PC, is to that in the Direction BP produced, as the Co-sine of the Angle CPH to the Co-sine of the Angle BPG, *(by the preceding Lemma)*; and, since these Celerities, when the

the Sum of CP and BP is a *Minimum*, must be equal *, it follows, therefore, that the said Angles CPH and BPG, as well as their Co-sines, will in that Circumstance become equal to each other; and consequently APC also equal to APB. From whence it appears, that (take AG what you will) the Sum of the three Lines, AP, BP, and CP, cannot be the least possible when the Angles APB and APC are unequal. And, by the same Argument, it also appears that their Sum cannot be the least possible, when the Angles BPA and BPC are unequal: Therefore, their Sum must be the least possible, when all the three Angles about the Point P are equal to one another; provided the Case will admit of such an Equality, or that no one of the Angles of the Triangle ABC is equal to, or greater than $\frac{1}{3}$ of 4 Right Angles (for otherwise, the Point P will fall in the obtuse Angle): Hence this

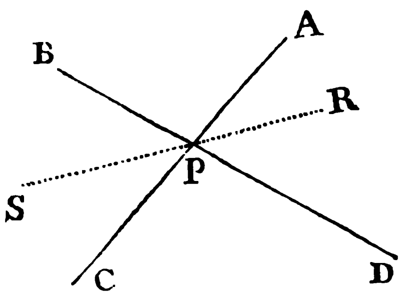


* Art. 2 and 22.

CONSTRUCTION.

Describe, upon BC, a Segment of a Circle, to contain an Angle of 120°, and let the whole Circle BCQ be compleated; and from A, to the Middle (Q) of the Arch BQC, draw AQ intersecting the Circumference of the Circle in P; which will be the Point required. For, the Angles BPQ and CPQ, standing upon the equal Arches BQ and CQ, have their Complements APB and APC equal to each other; and therefore, the Angle BPC being 120° (by Construction) each of the said

ſaid Angles APB, APC, will, likewiſe be 120 Degrees.



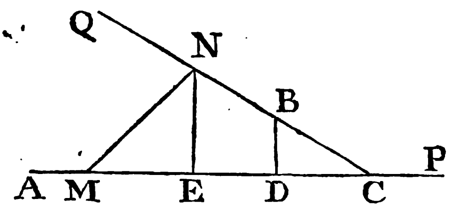
After the ſame Manner, it will appear that the Sum of all the Lines AP, BP, CP, &c. drawn from any Number of given Points A, B, C, &c. to meet in another Point P, will be the leaſt poſſible, when the Co-ſines of the Angles RPA, RPB, RPC, &c. that the ſaid Lines make with any other Line RS, paſſing through the Point of Concourſe, deſtroy each other: Which will be when all the Angles APB, BPC, CPD, &c. are equal, in all Caſes where the Poſition of the given Points will admit of ſuch an Equality. But, if the Number of given Points be four, the required Point will be in the Interſection of the two Right-lines joining the oppoſite Points: For, ſuppoſing APC and BPD to be continued Right-lines, the Co-ſine of RPA will be equal and contrary to that of RPC, and that of RPB equal and contrary to that of RPD.

EXAMPLE XIII.

37. *If two Bodies move at the ſame Time, from two given Places* A *and* B, *and proceed uniformly from thence in given Directions,* AP *and* BQ, *with Celerities in a given Ratio; it is propoſed to find their Poſition, and how far each has gone, when they are the neareſt poſſible to each other.*

Let M and N be any two cotemporary Poſitions of the Bodies, and upon AP let fall the Perpendiculars NE and BD; alſo let QB be produced to meet AP

in



in C, and let MN be drawn: Moreover, let the given Celerity in BQ be to that in AP, as n to m, and let AC, BC, and CD, (which are alſo given) be denoted by a, b, and c reſpectively, and make the variable Diſtance CN$=x$: Then, by reaſon of the parallel Lines NE and BD, we ſhall have b (CB) : x (CN) :: c (CD) : CE$=\frac{cx}{b}$. Alſo, becauſe the Diſtances, BN and AM, gone over in the ſame Time, are as the Celerities, we likewiſe have, $n : m :: x - b$ (BN) : AM $=\frac{mx - mb}{n}$, and conſequently CM (AC—AM)$=a+\frac{mb}{n}-\frac{mx}{n}=d-\frac{mx}{n}$, (by writing $d=a+\frac{mb}{n}$). Whence MN^2 ($=CM^2+CN^2-CM\times 2CE$) will alſo be found $=\overline{d-\frac{mx}{n}}^2+x^2-\overline{d-\frac{mx}{n}}\times\frac{2cx}{b}=d^2-\frac{2dmx}{n}+\frac{m^2x^2}{n^2}+x^2-\frac{2cdx}{b}+\frac{2cmx^2}{nb}$; whoſe Fluxion $-\frac{2dm\dot{x}}{n}+\frac{2m^2x\dot{x}}{n^2}+2x\dot{x}-\frac{2cd\dot{x}}{b}+\frac{4cmx\dot{x}}{nb}$ being made $=0$ (becauſe MN is to be a *Minimum*) we get $-bdmn+m^2bx+n^2bx-n^2cd+2mncx=0$; and conſequently $x=\frac{mnbd+n^2cd}{m^2b+n^2b+2mnc}=\frac{nd\times\overline{mb+nc}}{b\times\overline{m^2+n^2}+2mnc}$; from whence BN, AM, and MN are alſo given.

The

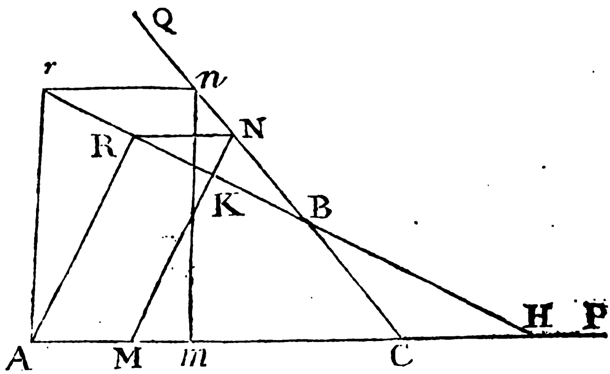
The same otherwise.

Because the relative Celerities of the two Bodies, at M and N, in the Direction of the Line MN (produced) are truly expressed by $\frac{\text{Co-sine M}}{\text{Radius}} \times m$, and $\frac{\text{Co-s.N}}{\text{Rad.}} \times n$, respectively *; and as these Celerities, when the Distance MN is a *Minimum*, do become equal to each other †, it follows that, in this Circumstance, $m : n ::$ Co-s. N. : Co-s. M :: Secant of M : Secant of N *(by plane Trig.)*

* Art. 35.

† Art. 22.

Whence this Construction. Take CH to CB in the given Ratio of m to n, and draw HB; upon which



produced (if necessary) let fall the Perpendicular AR; draw RN parallel to AH, meeting CQ in N; lastly, draw NM parallel to AR, and it will give the Position required. For, first, it is plain, because AM (RN) : BN (:: CH : CB) :: $m : n$, that M and N are cotemporary Positions: It is likewise plain, that RN and BN will be Secants of the Angles KNR (CMN) and KNB (CNM) to the Radius NK; because the Angle NKR (=ARK) is a Right one. Which Lines or Secants are in the proposed Ratio of m to n, as has been already shewn.

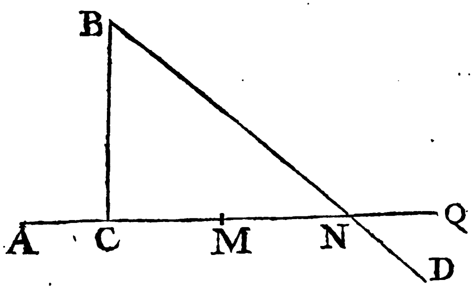
4

But

But the ſame Solution may be, yet, otherwiſe derived, independent of Fluxions, from Principles intirely geometrical. For, let m and n be any two cotemporary Poſitions at Pleaſure, and let CH (as before) be to CB, as the Celerity in AP to that in CQ; moreover, let nr, parallel to AP, be drawn, meeting HB produced in r, and let A, r be joined. Then, ſince CB : CH :: Bn : nr (*by ſim. Triangles*) and CB : CH :: Bn : Am, (*by Hyp.*) it follows, that nr and Am, (which are parallel) will alſo be equal to each other; and therefore Ar and mn, likewiſe equal and parallel. But Ar is the leaſt poſſible when perpendicular to Hr. Whence the Solution is manifeſt.

EXAMPLE XIV.

38. *Let the Body* M *move, uniformly, from* A *towards* Q, *with the Celerity* m, *and let another Body* N *proceed from* B, *at the ſame time, with the Celerity* n. *Now it is propoſed to find the Direction* (BD) *of the latter, ſo that the Diſtance* MN *of the two Bodies, when the latter arrives in the Way or Direction* AQ *of the former, may be the greateſt poſſible.*



Let BC be perpendicular to AQ, and make AC $= a$, BC$=b$, and BN$=x$. Therefore, if the Poſition M be ſuppoſed cotemporary with N, we ſhall have n : m :: x : AM $=\frac{mx}{n}$; whence CM$=\frac{mx}{n}-a$, and conſequently

ſequently MN (CN—CM) $=\sqrt{x^2-b^2}-\frac{mx}{n}+a$; whereof the Fluxion being taken, and made $=0$, we get $\frac{x}{\sqrt{x^2-b^2}}=\frac{m}{n}$; therefore $x=\frac{mb}{\sqrt{m^2-n^2}}$, and CN $(\sqrt{x^2-b^2})=\frac{nb}{\sqrt{m^2-n^2}}$: Whence, $m:n$ (:: BN : CN) :: Radius : Co-ſine N. The ſame Concluſion is otherwiſe derived, thus,

Let the Right-line BD be ſuppoſed to revolve about the Point B, as a Center, with a Motion ſo regulated, that the intercepted Part thereof BN may increaſe with the uniform Celerity n: Then, the Celerity with which CN is increaſed being $=\frac{n\times Radius}{Co\text{-}ſine\ N}$ *, this Expreſſion, when MN is a *Maximum*, muſt, conſequently, be equal to (m) the Velocity of the other Body † M; and therefore $m:n$ :: Radius : Co-ſine N, as before.

* Art. 35.

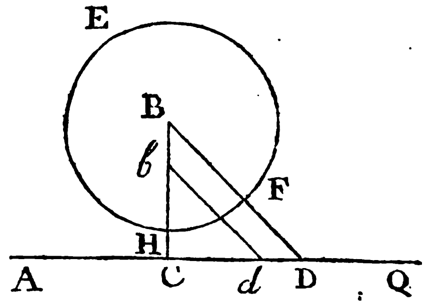
† Art. 22.

EXAMPLE XV.

39. *Suppoſing a Ship to ſail from a given Place* A, *in a given Direction* AQ, *at the ſame time that a Boat, from another given Place* B, *ſets out in order (if poſſible) to come up with her, and ſuppoſing the Rate at which each Veſſel runs to be given; it is required to find in what Direction the latter muſt proceed, ſo that, if it cannot come up with the former, it may, however, approach it as near as poſſible.*

Let the Celerity of the Ship be to that of the Boat in the given Ratio of m to n; alſo let D and F be the Places of the two Veſſels when neareſt poſſible to each other, and, from the Center B, through F, ſuppoſe the Circumference of a Circle to be deſcribed. Then (the Diſtance DF being the leaſt poſſible), the Point F muſt be in the Right-line (DB) joining the Point D and the Center

Center B; because no other Point in the whole Periphery, at which the Boat from B might arrive in the same time, is so near to D as that wherein the Line DB intersects the said



Periphery.—But now, to get an Expression for DF, in algebraic Terms, let BC be perpendicular to AQ, and make AC $=a$, BC $=b$, and CD $=x$; and then BD ($\sqrt{BC^2+CD^2}$) will be $=\sqrt{b^2+x^2}$; moreover, because $m:n::AD(a+x):BF$, you will have $BF=\frac{na+nx}{m}$, and consequently, $DF=\sqrt{b^2+x^2}-\frac{na+nx}{m}$; whose Fluxion, $\frac{x\dot{x}}{\sqrt{b^2+x^2}}-\frac{n\dot{x}}{m}$, being made $=0$, we find $x=\frac{nb}{\sqrt{m^2-n^2}}$; whence the Direction BD is known: And, if the Value of x, thus found, be substituted in that of DF, (found above) we shall have $DF=\frac{b\sqrt{m^2-n^2}-na}{m}$; whence the Position of F is known. And from which it is observable, that, as DF must be a *real*, *positive* Quantity (by the Question) this Method of Solution can only obtain when m is greater than n, and $b\sqrt{m^2-n^2}$, also greater than na: For in all other Cases the Boat will be able to come up with the Ship.

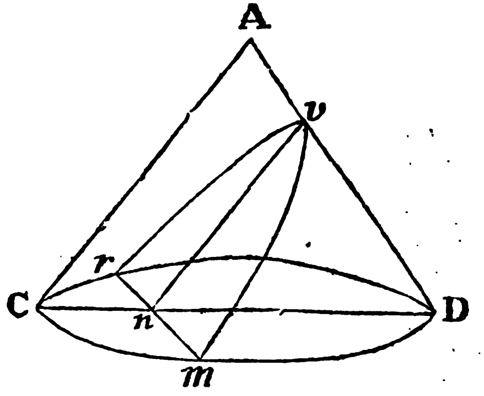
The same otherwise.

Let the Radius of the Circle EFH be conceived to increase uniformly, with the Celerity n, whilst the Point D moves

D moves uniform along AQ, with the Celerity *m*: Then, the Celerity at D, in the Direction of BD produced, being $=\frac{m\times Co\text{-}sine\ D}{Radius}$, the relative Celerity with which the Point D recedes from the Periphery of the said variable Circle, will be universally expressed by $\frac{m\times Co\text{-}sine\ D}{Radius}-n$; which being $=0$, when DF is a *Minimum*, we have in this Case $m\times Co\text{-}sine\ D=n\times Radius$, and consequently $m : n :: Radius : Co\text{-}sine$ D. Therefore, if, at C, a right-angled Triangle C*bd* be constituted, whose Base C$d=n$, and its Hypothenuse $db=m$, and parallel to the latter you draw BD, it will be the Direction required: In which, if there be taken BF, a Fourth-proportional to *m*, *n*, and AD, you will also have the Position required.

EXAMPLE XVI.

40. *To determine the greatest Parabola that can be formed by cutting a given Cone* ACD.



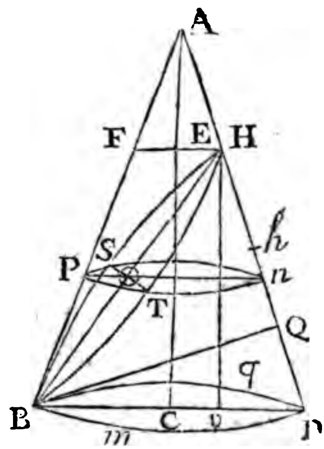
Let *nv*, parallel to CA, be the Axis of the Parabola *rvm*, and *rm* the Base (or Ordinate) thereof; putting DC

DC$=a$, CA$=b$, and D$n=x$; then, because of the parallel Lines, it will be, $a : b :: x : \frac{bx}{a} = nv$: Moreover, by the Property of the Circle, we have rn^2 ($= nm^2 =$ D$n\times$Cn) $= ax - x^2$, and consequently rm $2\sqrt{ax-x^2}$; which multiplied by $\frac{2}{3}\times\frac{bx}{a}$ (because every Parabola is $\frac{2}{3}$ of a Parallelogram of the same Base and Altitude) gives $\frac{4bx}{3a}\sqrt{ax-x^2}$ for the Content of the Parabola: Whose Fluxion, or that of $ax^3 - x^4$ * being put equal to Nothing; we find $x = \frac{3a}{4}$: Whence $nv =$ $\frac{3}{4}\times$AC, $rm =$ CD$\times\sqrt{\frac{3}{4}}$, and the Area of the greatest, or required, Parabola $=$ AC$\times$CD$\times\frac{\sqrt{3}}{4}$.

* Art. 26.

EXAMPLE XVII.

41. *To determine the greatest Ellipsis* BTES *that can be formed by cutting a given Cone* ABD.



Let BE be the greater, and TS the lesser, Axis of the Ellipsis BTES, considered as variable by the Motion of (the End of the Transverse) E, along the Line AD; moreover let Ev be parallel to AC the Axis of the Cone, meeting the Diameter BD in v, and let the Diameters EF and np be parallel to BD; whereof the latter np is supposed to pass through O the Center of the Ellipsis: Then, putting AC$=a$, CD$=b$, and C$v=x$, we shall have B$v=b+x$; also, because of the parallel Lines we have CD (b) : CA (a) :: Dv $(b-x)$: $\frac{a\times\overline{b-x}}{b}=Ev$; whence

$$BE\ (\sqrt{Bv^2+Ev^2}) = \frac{\sqrt{b^2\times\overline{b+x}^2+a^2\times\overline{b-x}^2}}{b}.$$

Furthermore, since the Triangles EOn, EBD, and BOp, BEF are equiangular, and EO (=BO) $=\frac{1}{2}$BE, we likewise have O$n=\frac{1}{2}$BD$=b$, and O$p=\frac{1}{2}$EF = Cv $=x$; and consequently O$n\times$Op (=OT^2, *by the Property of the Circle*) $=bx$; whence ST $=2\sqrt{bx}$, and

therefore $BE\times ST=\frac{\sqrt{b^2\times\overline{b+x}^2+a^2\times\overline{b-x}^2\times 4bx}}{b}$.

Now the Area of any Ellipsis being in a constant Ratio to the Rectangle of its greater and lesser Axes (namely as 3,14159, &c. to 4) the last general Expression must therefore be a *Maximum*, when the Area is so; and therefore its Fluxion, or that of $b^2x\times\overline{b+x}^2+a^2x\times\overline{b-x}^2$ $(=b^4x+2b^3x^2+b^2x^3+a^2b^2x-2a^2bx^2+a^2x^3)$ equal to Nothing *; that is, $b^4\dot{x}+4b^3x\dot{x}+3b^2x^2\dot{x}+a^2b^2\dot{x}-4a^2bx\dot{x}+3a^2x^2\dot{x}=0$:

* Art. 22.

Whence $x^2-\frac{4bx\times\overline{a^2-b^2}}{3a^2+3b^2}=-\frac{b^2}{3}$, and $x=\frac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; from which the Ellipsis is known.

But it is observable, that, when $a^4-14a^2b^2+b^4$ is negative, this Solution fails, because the Square Root of a negative Quantity is to be extracted. Therefore, to determine the Limit, put $a^4-14a^2b^2+b^4=0$; then, by ordering the Equation, you will get $a^2=b^2\times\overline{7+\sqrt{48}}$, and $a=b\times\overline{2+\sqrt{3}}$; and therefore $a:b::2+\sqrt{3}:1$. Hence, if the Ratio of AC to CD be not

 greater

greater than that of $2+\sqrt{3}$ to 1, or (which comes to the same thing) if the Angle DAC be not less than 15 Degrees, the Fluxion of the Ellipsis can never become equal to Nothing; but the Ellipsis itself will increase continually, from the Vertex till it coincides with the Base of the Cone; and therefore is greater at the Base than in any other Position.

But it is further to be observed, that this Problem is confined to, yet, narrower Limits. For, either the Ellipsis will increase, continually, from the Vertex, to the Base, of the Cone, (which is shewn to be the Case when the Angle DAC is greater than 15°) or else it will increase till the Point E arrives at a certain Position H, and afterwards decrease to another certain Position *b*, and then increase again till it coincides with the Base of the Cone, (for it must always increase again before it coincides with the Base, because, after the Point E is got below the Perpendicular BQ, both the Axes of the Ellipsis increase at the same time).

The same thing also appears from the foregoing Equation $x = \frac{2b \times \overline{a^2 - b^2} \pm b\sqrt{a^4 - 14a^2b^2 + b^4}}{3a^2 + 3b^2}$; whose two Roots express the two Values of x (or Cv) at the Times of the *Maximum* (at H) and its succeeding *Minimum* (at *b*). Hence it is manifest, that the Ellipsis may admit of a *Maximum* between the Vertex of the Cone and the Perpendicular BQ, and yet, that *Maximum* be less than the Base of the Cone, unless the foresaid Angle DAC be so much less than 15° (above found) that the Increase from *b* to D, be less than the Decrease from H to *b*. Now therefore, to determine the exact Limit, let the foresaid Increment and Decrement be supposed equal to each other, or, which is the same in Effect, let the Ellipsis BTESB = the Circle BqDm, or BE×ST=BD², that is, let

$$\frac{\sqrt{\overline{b^2 \times \overline{b+x}^2 + a^2 \times \overline{b-x}^2} \times 4bx}}{b} = 4b^2$$: From which

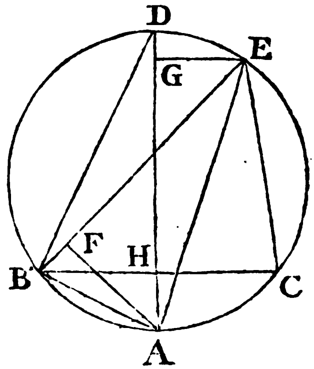
Equation you will get $a^2 = \frac{b^2}{x} \times \frac{4b^3 - b^2x - 2bx^2 - x^3}{\overline{b-x}|^2}$ $= \frac{b^2}{x} \times \frac{4b^2+3bx+x^2}{b-x}$: Moreover, from the Equation $b^4\dot{x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$, (given above) you will, again, get $a^2 = \frac{b^2 \times \overline{b^2+4bx+3x^2}}{-b^2+4bx-3x^2}$ $= \frac{b^2 \times \overline{b^2+4bx+3x^2}}{\overline{b-x} \times \overline{3x-b}}$: Whence, by comparing these equal Values, there arises $\frac{4b^2+3bx+x^2}{x} = \frac{b^2+4bx+3x^2}{3x-b}$, which, ordered, gives $x^2+2bx-b^2=0$, and therefore $x = b\sqrt{2} - b$.

Moreover, $\frac{a^2}{b^2}$ being $= \frac{4b^2+3bx+x^2}{bx-x^2}$, if b^2-2bx be substituted herein for, its Equal, x^2, it will become $\frac{a^2}{b^2} = \frac{5b^2+bx}{bx-x^2} = \frac{5b+x}{3x-b} = \frac{5b+b\sqrt{2}-b}{3b\sqrt{2}-3b-b} = \frac{4+\sqrt{2}}{-4+3\sqrt{2}}$ $= \frac{\overline{4+\sqrt{2}} \times \overline{4+3\sqrt{2}}}{\overline{-4+3\sqrt{2}} \times \overline{4+3\sqrt{2}}} = \frac{22+16\sqrt{2}}{2} = 11 + 8\sqrt{2}$. Hence we have, $1 : \sqrt{11+8\sqrt{2}} :: b\ (DC) : a\ (AC)$ :: Radius to the Tangent of the Angle ADC $= 78° 3'$: Whose Complement DAC $= 11° 57'$, is the least Limit possible. Therefore, unless the Angle which the slant Side makes with the Axis be less than $11° 57'$, the greatest Ellipsis will be less than the Base of the Cone.

EXAMPLE XVIII.

42. *Of all Triangles, having the same given Perimeter, and inscribed in the same given Circle; to determine the greatest.*

Let the Diameter DA bisect the Base BC of the required Triangle BEC in H, draw AE, AB and BD; and draw AF perpendicular to BE, and GE, parallel to BC,



BC, meeting AD in G: Then, putting AD $= a$, half the given Perimeter of the Triangle $= b$, and DH $= y$; we have BH $= \sqrt{ay - y^2}$, and therefore EF $= b - \sqrt{ay - y^2}$ [a]. Moreover DH (y) : AD (a) :: DB^2 : DA^2 :: EF^2 $\left(\overline{b - \sqrt{ay - y^2}}^{\,2}\right)$: EA^2 $= \frac{a}{y} \times \overline{b - \sqrt{ay - y^2}}^{\,2}$;

therefore AG $\left(\frac{AE^2}{AD}\right) = \frac{\overline{b - \sqrt{ay - y^2}}^{\,2}}{y}$, and HG $=$ (AG—AH$=$) $\frac{b^2 - 2b\sqrt{ay - y^2}}{y}$; whence the Area of the Triangle BEC (BH×HG) $= \frac{b^2\sqrt{ay - y^2}}{y} - 2ba + 2by$, whose Fluxion $2b\dot{y} - \frac{\frac{1}{2}ab^2\dot{y}}{y\sqrt{ay - yy}}$ being put $= 0$, gives $y\sqrt{ay - yy} = \frac{1}{4}ba$; whence y, and from thence the Sides of the Triangle may be determined.

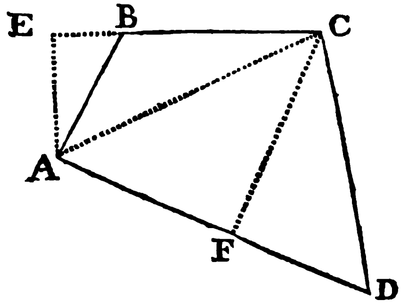
EXAMPLE XIX.

43. *To determine the greatest Area that can be contained under four given Right-lines.*

Though it is demonstrable from common Geometry that the Area will be a *Maximum*, when the Trapezium ABCD, formed by the given Lines, may be inscribed in a Circle [b], yet I shall here give the Solution from the Principles of Fluxions, (whose Uses I am now

[a] By Prop. 13. Page 62. *Elem. Trig.*
[b] See Page 117 of *Elem. Geometry.*

illuſtrating). In order to which, let the Diagonal AC be drawn, and upon CB and AD let fall the Perpendiculars AE and CF; putting AB$=a$, BC$=b$, CD$=c$, DA$=d$, BE$=x$, and DF $=y$: Then AE being $=\sqrt{a^2-x^2}$, and CF$=\sqrt{c^2-y^2}$, the Area of the Trapezium ($\frac{1}{2}$BC × AE + $\frac{1}{2}$AD×CF) will be$=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{1}{2}d\sqrt{c^2-y^2}$;



* Art. 22. and its Fluxion $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{a^2-x^2}}-\frac{\frac{1}{2}dy\dot{y}}{\sqrt{c^2-y^2}}=0$ *; and therefore $\frac{-dy\dot{y}}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$. Moreover, ſince b^2+a^2+2bx ($=$AC^2) $=d^2+c^2-2dy$, by taking the Fluxion thereof, we have $2b\dot{x}=-2d\dot{y}$, or $-d\dot{y}=b\dot{x}$; which, ſubſtituted for $-d\dot{y}$ in the foregoing Equation, gives $\frac{b\dot{x}y}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$, and $\frac{y}{\sqrt{c^2-y^2}}=\frac{x}{\sqrt{a^2-x^2}}$; and conſequently, $\sqrt{c^2-y^2}$ (CF) : y (DF) :: $\sqrt{a^2-x^2}$ (AE) : x (BE): From which it appears that the Triangles DCF and ABE are ſimilar, and that (D+ABC being$=$ 2 Right Angles) the Trapezium may be inſcribed in a Circle; but this by the Bye. We are now to get an Expreſſion for the Area in known Terms, and in Order thereto we have $b^2+a^2+2bx=dd+c^2-2dy$, $y=\frac{cx}{a}$, and CF$=\frac{c\sqrt{a^2-x^2}}{a}$ (becauſe AB : BE :: DC : DF, &c.): Therefore, by Subſtitution, $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, and the Area ($\frac{1}{2}$BC × AE +$\frac{1}{2}$AD

$+\frac{1}{2}AD\times CF)=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{cd}{2a}\sqrt{a^2-x^2}=\frac{ab+cd}{2a}\sqrt{a^2-x^2}$; and therefore the Square thereof $=$

$$\frac{\overline{ab+cd}^2}{4a^2}\times\overline{a^2-x^2}=\frac{\overline{ab+cd}^2}{4a^2}\times\overline{a+x}\times\overline{a-x}=\frac{\overline{ab+cd}^2}{4}$$

$\times\overline{1+\frac{x}{a}}\times\overline{1-\frac{x}{a}}$. But since $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, we have $\frac{x}{a}=\frac{d^2+c^2-b^2-a^2}{2ab+2cd}$, $1+\frac{x}{a}=1+\frac{dd+c^2-b^2-a^2}{2ab+2cd}=\frac{2ab+2cd+dd+c^2-b^2-a^2}{2ab+2cd}=$

$\frac{\overline{d+c}^2-\overline{b-a}^2}{2ab+2cd}$; and $1-\frac{x}{a}=\frac{2ab+2cd-dd-c^2+b^2+a^2}{2ab+2cd}$

$=\frac{\overline{b+a}^2-\overline{d-c}^2}{2ab+2cd}$; and consequently, the Square of the

$$\text{Area}=\frac{\overline{ab+cd}^2}{4}\times\frac{\overline{d+c}^2-\overline{b-a}^2}{2ab+2cd}\times\frac{\overline{b+a}^2-\overline{d-c}^2}{2ab+2cd}$$

$=\frac{\overline{\overline{d+c}^2-\overline{b-a}^2}\times\overline{\overline{b+a}^2-\overline{d-c}^2}}{16}$ which (because the Difference of the Squares of any two Quantities is equal to a Rectangle under their Sum and Difference) will also be $=\frac{\overline{d+c+b-a}\times\overline{d+c-b+a}\times\overline{b+a+d-c}}{4}\times$

$\frac{\overline{b+a-d+c}}{4}=\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-a}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-b}$

$\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-d}$. Whence it appears, that, if from $\frac{1}{2}$ the Sum of all the four Sides each particular Side be substracted, the continual Product of the Remainders will be the Square, or second Power, of the Area.

From this Theorem, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is deduced, as a Corollary: For, making $a=0$,

$a=0$, the Trapezium becomes a Triangle, and the second Power of its Area$=\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b-b}$ $\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b-d}$: Which, in Words, is the common Rule.

EXAMPLE XX.

44. *To find the greatest Value of y in the Equation* $a^4x^2=\overline{xx+yy}^3$.

By putting the whole Equation into Fluxions, &c. we have $2a^4x\dot{x}=\overline{2x\dot{x}+2y\dot{y}}\times3\times\overline{x^2+y^2}^2$; which in the required Circumstance, when $\dot{y}=0$ *, becomes $2a^4x\dot{x}$ $=6x\dot{x}\times\overline{x^2+y^2}^2$; whence $x^2+y^2=\frac{a^2}{\sqrt{3}}$, and $\overline{x^2+y^2}^3$ $=\frac{a^6}{3\sqrt{3}}$: But, by the given Equation $\overline{x^2+y^2}^3=a^4x^2$; consequently $a^4x^2=\frac{a^6}{3\sqrt{3}}$, and therefore $x=$ $a\sqrt{\frac{1}{3\sqrt{3}}}$; whence $y^2\left(=\frac{a^2}{\sqrt{3}}-x^2\right)=\frac{2a^2}{3\sqrt{3}}$, and $y=a\sqrt{\frac{2}{3\sqrt{3}}}$.

* Art. 22.

The same otherwise.

Since $\overline{xx+yy}^3$ is given $=a^4x^2$, we have $x^2+y^2=$ $a^{\frac{4}{3}}\times x^{\frac{2}{3}}$, and therefore $y^2=a^{\frac{4}{3}}\times x^{\frac{2}{3}}-x^2$; whose Fluxion, $\frac{2}{3}a^{\frac{4}{3}}\times x^{-\frac{1}{3}}\dot{x}-2x\dot{x}$, being put $=0$, we also get $\frac{a^{\frac{4}{3}}\times x^{-\frac{1}{3}}}{3}$ $=x$; whose Cube is $\frac{a^4\times x^{-1}}{27}=x^3$, or $\frac{a^4}{27x}=x^3$; whence $27x^4=a^4$, and consequently $x=a\sqrt{\frac{1}{3\sqrt{3}}}$, the same as before.

45. When

45. When, in the general Expression, whose *Maximum* or *Minimum* is sought, there are two or more indeterminate Quantities, independent of each other, their respective Values, in the required Circumstance, will be determined, by making them flow, one by one, while the others are supposed invariable; as in the following

EXAMPLE XXI.

Wherein it is proposed to find three such Values of x, y, and z, as shall make the Value of $\overline{b^3-x^3}\times\overline{x^2z-z^3}\times\overline{xy-y^2}$ *the greatest possible.*

First, considering y as variable, and the rest constant, we have $x\dot{y}-2y\dot{y}=0$ *; whence $y=\frac{1}{2}x$, and $xy-y^2=\frac{1}{4}x^2$. By making z variable, we have $x^2\dot{z}-3z^2\dot{z}=0$; whence $z=\frac{x}{\sqrt{3}}$, and $x^2z-z^3=\frac{2x^3}{3\sqrt{3}}$. Now let these Values of $xy-y^2$ and x^2z-z^3 be substituted in the given Expression, and it will become $\frac{x^2}{4}\times\frac{2x^3}{3\sqrt{3}}\times\overline{b^3-x^3}=\frac{b^3x^5-x^8}{6\sqrt{3}}$; therefore $5b^3x^4\dot{x}-8x^7\dot{x}=0$: Whence $x=\frac{1}{2}b\times\sqrt[3]{5}$, $y\left(=\frac{1}{2}x\right)=\frac{1}{4}b\times\sqrt[3]{5}$, and $z\left(=\frac{x}{\sqrt{3}}\right)=\frac{1}{2}b\times\frac{\sqrt[3]{5}}{\sqrt{3}}$.

* Art. 22.

The Reason of the foregoing Process is obvious: For, if the Fluxion of the given Expression, when any one of the indeterminate Quantities is made variable, be not equal to Nothing, that Expression may become greater, without altering the Values of the rest, which are considered as constant †: And therefore cannot be the greatest possible, unless the said Fluxion is equal to Nothing.

† Art. 22.

EXAMPLE XXII.

46. *To determine the different Values of x, when that of* $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$ *becomes a* Maximum *or* Minimum.

The Fluxion of the given Expreſſion being *(as uſual)* put equal to Nothing, we have $12x^3-84ax^2+168a^2x-96a^3=0$, or $x^3-7ax^2+14a^2x-8a^3=0$: From whence (by the Method of Diviſors) we get $x-a=0$, $x-2a=0$, or $x-4a=0$: Therefore, the Roots of the Equation, or the three Values of x, are a, $2a$, and $4a$.

SCHOLIUM.

47. It appears, from the laſt Example, that a Quantity may admit of as many *Maxima* and *Minima* (according to the Meaning of the Definition *) as there are poſſible Roots in the Equation, ariſing from aſſuming its Fluxion equal to Nothing. Now to know which of thoſe Roots point out a *Maximum*, and which a *Minimum*; find whether the Value of the ſaid Fluxion, a little before it becomes equal to Nothing, be poſitive or negative; if *poſitive*, the ſucceeding Root gives a *Maximum*; but if *negative*, a *Minimum*: The Reaſon of which is extremely obvious; becauſe ſo long as any Quantity increaſes, its Fluxion is poſitive, but when it decreaſes the Fluxion is negative.

* Art. 22.

As an Example hereof, let the Quantity $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$, be again reſumed; whoſe Fluxion is $12\dot{x}\times\overline{x^3-7ax^2+14a^2x-8a^3}=12\dot{x}\times\overline{x-a}\times\overline{x-2a}\times\overline{x-3a}$: Whereof the Value, before it becomes equal to Nothing, the firſt time (or before $x=a$) being negative (becauſe the Product of three negative Factors is negative) its firſt Root (a) therefore indicates a *Minimum*: Whence we may conclude, without conſidering farther, that the ſecond Root $(2a)$ gives a *Maximum*, and the third $(4a)$ another *Minimum*. But, if you

you would know whether the first or third Root gives the lesser Value of the two; it is but substituting in the given Quantity, which will come out $48b^4—37a^4$, and $48b^4—64a^4$ respectively; therefore the latter is the lesser, and the very least Value the proposed Expression can admit of.

When all the Roots prove impossible, the Quantity proposed (as its Fluxion can never become $=0$) must either increase, or decrease, continually; and therefore can neither admit of a *Maximum nor a Minimum.*

Moreover, it may so happen, that the Roots are possible, the Fluxion $=0$, and yet the Quantity itself be neither a *Maximum* nor a *Minimum* in that Circumstance.

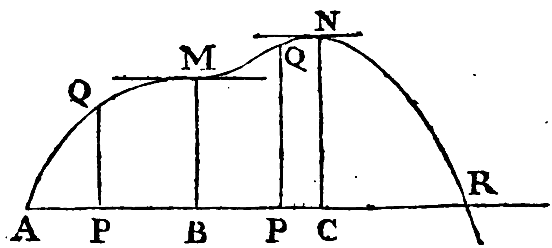
For let us, again, suppose the Point *n* to move after *m*, as in the general Illustration, (*vid. Art.* 22.) only let the Velocity of *n (in the first Case)* increase no longer than 'till it arrives at D; after which let it again decrease: Then, though the Fluxion of the Distance *mn* is Nothing, at the Position CD, yet the Distance itself will not be a *Maximum*; because *n* (having afterwards, as well as before, a less Velocity than *m)* will still continue to lose ground.—In the same manner the Matter may be explained with regard to a *Minimum.* And it is evident, that these Cases will always happen when the Fluxion of the given Quantity is of the same Denomination (with regard to positive and negative) both before and after, it becomes equal to Nothing: Which, by the Rules of common Algebra, is known to be when the Equation admits of an even Number of equal Roots.—An Example hereof, however, may not be improper.

Let then the Quantity proposed be $24a^3x — 30a^2x^2 + 16ax^3—3x^4$; whose Fluxion is $24a^3\dot{x}—60a^2x\dot{x} + 48ax^2\dot{x}—12x^3\dot{x} = 12\dot{x}\times\overline{a—x}\times\overline{a—x}\times\overline{2a—x}$: Which being made $=0$, it appears that the two least Roots are equal. Therefore there is neither a *Maximum* nor *Minimum* when $x=a$ (because whether x be taken a little less, or a little greater, than a, the Value of the Fluxion will

will ſtill be affirmative.) The greateſt Root, however, not being affected with another equal one, indicates a *Maximum*, according to the Rule above preſcribed.

To render what has been obſerved above ſtill more conſpicuous, let the given Expreſſion, $24a^3x - 30a^2x^2 + 16ax^3 - 3x^4$, be repreſented by the variable Ordinate PQ of the Curve AQMNR, whoſe Abſciſſa AP is (as uſual) denoted by x.

Then, whilſt ($12\dot{x} \times \overline{a-x} \times \overline{a-x} \times \overline{2a-x}$) the Fluxion of the Ordinate continues poſitive, (or 'till x becomes $=a=$AB) the Ordinate itſelf will increaſe: But at the Poſition BM it becomes ſtationary (if I may be allowed the Expreſſion) the Fluxion being then $=0$. After which, the Fluxion being again affirmative, the Ordinate will again increaſe, till x becomes $= 2a$ ($=$ AC); when, the Fluxion becoming Nothing, (a ſe-

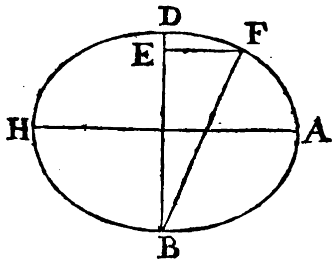


cond time,) and afterwards negative, CN will be a *Maximum:* Soon after which the Curve deſcends below its Axis, and continues to recede from it *in infinitum.*

Another Thing there is that ought to be regarded in the Solution of theſe Kinds of Problems, and that is, whether the *Maxima* or *Minima*, found by aſſuming the Fluxion $=0$, fall within the Limits preſcribed by the Nature of the Queſtion or Figure; which is often reſtrained by Conditions that do not enter into the algebraic Computation.

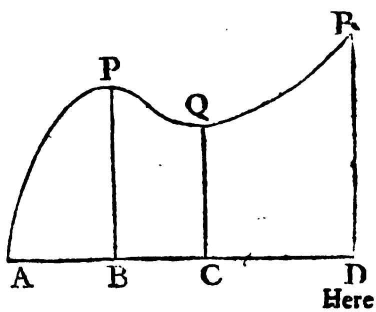
Thus, for Example; ſuppoſe it were required to find that Point (F) in a given Ellipſis ABHD which, of all others,

others, is the most remote from the Extreme B of the conjugate Axis BD.



Then, drawing FE parallel to the Transverse AH, and putting AH$=a$, BD $=b$, and BE$=x$, we have, by the Property of the Curve BF^2 ($=$BE$^2+$EF^2) $x^2 + \overline{bx-x^2} \times \frac{a^2}{b^2}$; from whence x is found $= \frac{\frac{1}{2}a^2b}{a^2-b^2}$. But, from the Nature of the Figure, the greatest Value that x ($=$BE) can possibly admit of is b($=$BD), therefore if the Relation of a and b be such, that $\frac{\frac{1}{2}a^2b}{a^2-b^2}$ is greater than b, this Solution is manifestly impossible.—— To determine the Limit, therefore, make $\frac{\frac{1}{2}a^2b}{a^2-b^2}=b$; then it will be found that $2b^2=a^2$. Whence the foregoing Solution can only obtain when 2BD^2 is equal to, or less than AH^2.

Again, it ought to be also considered whether the Value of x, found by the common Method, gives a less Quantity for the *Maximum*, and a greater for the *Minimum*, than will arise from the Extremes themselves by which x is limited.

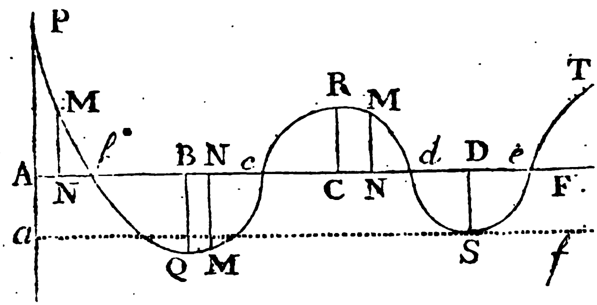


Thus, let it be required to determine the greatest and least Ordinates in a Curve, APR, whose Equation is $y^3=6a^2x-9ax^2+4x^3$, and whose greatest Abscissa AD is given equal $2a$.

Here

Here we shall, by taking the Fluxion, &c. have $x=\frac{1}{2}a$, or $x=a$: The former of which Values gives the corresponding Ordinate $BP=a\sqrt[3]{\frac{5}{4}}$; and the latter, $CQ=a$: But the first of these is not the greatest of all others, because the Extreme DR exceeds it, being $=2a$; nor is CQ the least possible, because the Ordinate at the other Extreme A is nothing at all.

Sometimes one, or more, of the Points Q, S, &c. determining the *Maxima* and *Minima*, will fall below the Axis AF, *(as in the annexed Figure)*. In which Case the corresponding Value of the general Expression, represented by the Ordinate, will be negative: But at the Points *b*, *c*, *d*, &c. where the Curve intersects the



Axis, it will be equal to Nothing: Whence (by the Bye) the Reason why the Roots of an Equation ($x^n-ax^{n-1}+b^2x^{n-2}\ldots+q^n=0$) are impossible by Pairs is evident. For, seeing A*b*, A*c*, A*d*, A*e*, &c. are the Roots of that Equation, or the different Values of x, when the Ordinate $x^n-ax^{n-1}+b^2x^{n-2}\ldots\ldots+q^n$ (MN) becomes equal to Nothing, it is plain, if PA, expressing the given Term q^n, be increased to P*a*, so that AF (then coinciding with *af*) may touch the Curve in S, the adjacent Roots A*d* and A*e* will then become equal;

equal; and if $\overset{n}{q}$ be farther increased, so that the Axis may fall wholly below the Curve, not only those two, but also the other Roots, Ab and Ac, will become impossible.

Various other Observations might be made, relating to the Limits of Equations, determined by these *Maxima* and *Minima*; but this being foreign to the Matter in hand, I shall content myself with one Remark more, *viz.*

Any Expression, which being put equal to Nothing, admits of two or more equal Roots, has as many succeeding Orders of Fluxions equal to Nothing, at the same time, as are expressed by the Number of those Roots minus one.

Thus, an Equation, having three equal Roots, has both its first and second Fluxions equal to Nothing, when the Fluent itself is equal to Nothing.

Hence we have another Way (besides that given above) to know when a Quantity may have its Fluxion equal to Nothing, and yet neither admit of a *Maximum* nor a *Minimum*: For, since this Circumstance always takes place when the Equation admits of an *even* Number of equal Roots (as has been already shewn) the Number of Orders of Fluxions, equal to Nothing, at the same time (including the First) must also be even.

Hence, also, we have an easy Method for discovering when some of the Roots of an Equation are equal; and, if so, what they are.

Thus, let $x^3 - 3ax^2 + 4a^3 = 0$ be propounded; whereof the Fluxion $3x^2\dot{x} - 6ax\dot{x}$ being assumed equal to Nothing, we find $x = 2a$; which will also be a Root of the given Equation, if it admits of two equal ones: To try it, therefore, I substitute $2a$ for x, and find it answers.

Again, let $8x^4 - 28ax^3 + 18a^2x^2 + 27a^3x - 27a^4 = 0$; whereof the first and second Fluxions being $32x^3\dot{x} - 84ax^2\dot{x} + 36a^2x\dot{x} + 27a^3\dot{x}$ and $96x^2\dot{x}^2 - 168ax\dot{x}^2 + 36a^2\dot{x}^2$, if the latter of them be assumed $=0$, x will be found $=\frac{7a}{8}\pm\sqrt{\frac{25a^2}{64}}=\frac{3a}{2}$, or $\frac{a}{4}$: One of which Quantities, if the Equation propofed admits of three equal Roots, will be the Value of each of them: By trying $\frac{3a}{2}$, it will be found to fucceed. Whence, by a well known Rule, the fourth Root (being $=\frac{28a}{8}-\frac{3a}{2}\times 3=-a$) is alfo given.

The Reafon of thefe Operations, as well as what is afferted above, may be thus demonftrated.

Let $\overline{r-x}\times\overline{r-x}$ &c. $\times\overline{A+Bx+Cx^2\ \&c.}=0$, be any Equation, having two or more equal Roots, reprefented, each, by r: Put $y=r-x$, and let the Number of the equal Roots be denoted by n; then, by Subftitution, we have $y^n\times\overline{A+B\times\overline{r-y}+C\times\overline{r-y}|^2\ \&c.}=0$; which, by expanding the Powers of $r-y$, and putting $a=A+Br+Cr^2$ &c. $b=B+2Cr+3Dr^2$ &c. will be further transformed to $y^n\times\overline{a-by+cy^2-dy^3}$ &c. $=0$: Whofe Fluxion $na\dot{y}y^{n-1}-\overline{n+1}\,.\,b\dot{y}y^n+\overline{n+2}\,.\,c\dot{y}y^{n+1}$ &c. is evidently equal to Nothing, when y, or its Equal $r-x$, is Nothing (provided n be greater than Unity). It is equally plain, that the fecond Fluxion $n.\overline{n-1}\,.\,a\dot{y}^2y^{n-2}-\overline{n+1}\,.\,nb\dot{y}^2y^{n-1}+\overline{n+2}.\overline{n+1}\,.\,c\dot{y}^2y^n$ &c. will alfo be equal to Nothing, in the fame Circumftance, if n be greater than 2. &c. &c.

Hence, univerfally, let the Number (n) of equal Roots be what it will, that of the Orders of Fluxions equal to Nothing, at the fame time, will be expreffed by that Number *minus* one, as was to be fhewn.

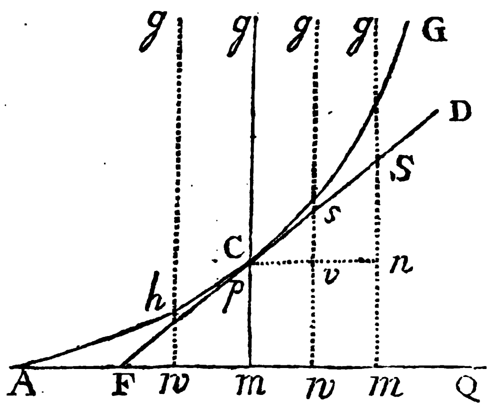
SECT.

SECTION III.

The Use of FLUXIONS *in drawing Tangents to Curves.*

ILLUSTRATION.

48. LET ACG be a Curve of any kind, and C the given Point from whence the Tangent is to be drawn.

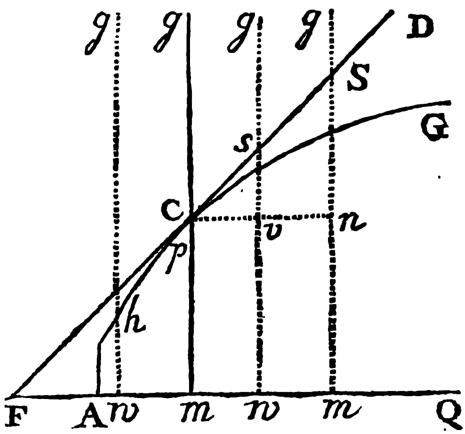


Conceive a Right-line *mg* to be carried along uniformly, parallel to itſelf, from A towards Q, and let, at the ſame time, a Point *p* ſo move in that Line, as to deſcribe, or trace out, the given Curve ACG: Alſo let *mm*, or C*n* (equal and parallel to *mm*) expreſs the Fluxion of A*m*, or the Celerity wherewith the Line *mg* is carried; and let *n*S expreſs the correſponding Fluxion of *mp*, in the Poſition *m*C*g*, or the Celerity of the Point *p*, in the Line *mg*. Moreover, through the Point C let the Right-line SF be drawn, meeting the Axis of the Curve (AQ) in F.

 Now,

Now, it is evident, if the Motion of *p*, along the Line *mg*, was to become equable at C, the Point *p* would be at S, when the Line itself had acquired the Position *m*S*g* (because, by Hypothesis, C*n* and *n*S express the Distances that might be described by the two uniform Motions in the same time).

And, if *wsg* be assumed to represent any other Position of that Line, and *s* the contemporary Position of the Point *p* (still supposing an equable Celerity of *p*); then the Distances C*v* and *vs*, gone over, in the same



time, by the two Motions, will, always, be to each other as the Celerities, or as C*n* to *n*S: Therefore, since C*v* : *vs* :: C*n* : *n*S (which is a known Property of similar Triangles) the Point *s* will, always, fall in the Right-line FCS: Whence it appears, that, if the Motion of the Point *p* along the Line *mg* was to become uniform at C, that Point would then move in the Right-line CS, instead of the Curve-line CG.

Now, seeing the Motion of *p*, in the Description of Curves, must, either, be an accelerated or a retarded one, let it be, first, considered as an accelerated one: In which Case the Arch CG will fall, *wholly*, above the Right-line CD (as in Fig. 1.) because the Distance of of the Point p from the Axis AQ, at the End of any given Time, is greater than it would be if the Acceleration was to ceafe at C; and, if the Acceleration had ceafed at C, the Point p would (it is proved) have been always found in the faid Right-line FS.

But if the Motion of the Point p be a retarded one, it will appear, by reafoning in the fame manner, that the Arch CG will fall wholly below the Right-line CD (as in Fig. 2.)

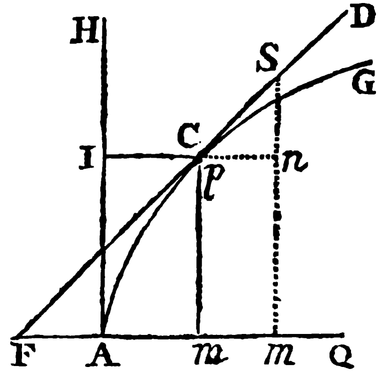
This being the Cafe, let the Line mg, and the Point p, along that Line, be now fuppofed to move back again, towards A and m, in the fame manner they proceeded from thence: Then, fince the Celerity of p (*Fig.* 1.) did before increafe, it muft now, on the contrary, decreafe; and, therefore, as p, at the End of a given Time, after repaffing the Point C, is not fo near to AQ, as it would have been, had the Velocity continued the fame as at C, the Arch Cb (as well as CG) muft fall wholly above the Right-line FCD. And, by the fame Method of arguing, the Arch Cb, *in the fecond Cafe*, will fall, *wholly*, below FCD: Therefore FCD, *in both Cafes*, is a Tangent to the Curve at the Point C: Whence, the Triangles FmC and CnS being fimilar, it appears, that the Sub-tangent mF is always a Fourth-proportional to (nS) the Fluxion of the ordinate (Cn), the Fluxion of the Abfciffa, and the Ordinate (Cm).

Otherwife.

49. Let ACG reprefent the propofed Curve, and let the Right-line FCD be a Tangent to it, at any Point C, meeting the Axis AQ (produced if neceffary) in F: Suppofe a Point p to move along the Curve, from A towards G, and let the abfolute Celerity thereof at C, in the Direction of the Tangent CD, or the Fluxion of the Line Ap fo generated *, be denoted by CS, any * Art. 2 and 5.
Part of the faid Tangent: Then, if AH, mp and mS be made perpendicular, and Ipn parallel, to AQ, the relative Celerities of that Point, in the Directions Cn and mC, wherewith Ip (=Am) and mp increafe in this

 Pofition,

* Art. 35. Position, will be truly expressed by Cn and nS *: But the Celerities by which Quantities increase are as the Fluxions of those Quantities: Therefore (CS being the Fluxion of the Curve-line Ap) Cn and nS are the corresponding Fluxions of the Abscissa Am and the Ordinate mp: And we have Sn : nC :: mC : mF, *the same as before.*

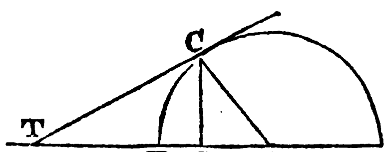


Hence, if the Abscissa Am be put $=x$, and the Ordinate $mp = y$, we shall have $mF = \frac{y\dot{x}}{\dot{y}}$: By means of which general Expression, and the Equation expressing the Relation between x and y, the Ratio of the Fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the Length of the Sub-tangent (mF) as in the following Examples.

EXAMPLE I.

50. *To draw a Right-line* CT, *to touch a given Circle* BCA, *in a given Point* C.

Let CS be perpendicular to the Diameter AB, and put AB $= a$, BS $= x$ and SC $= y$: Then, by the Property of the Circle, y^2 (CS^2) $=$ BS $\times$ AS($= x \times \overline{a-x}$) $= ax - x^2$; whereof



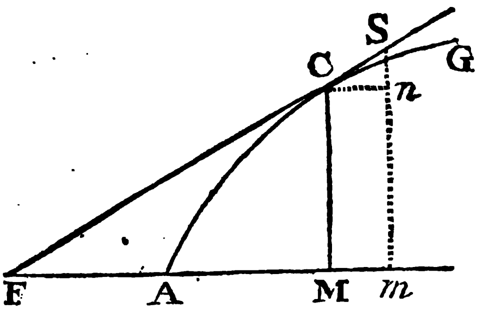
whereof the Fluxion being taken, in order to determine the Ratio of $\dot{x}$ and $\dot{y}$, we get $2y\dot{y} = a\dot{x} - 2x\dot{x}$; consequently $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a-2x} = \frac{y}{\frac{1}{2}a-x}$; which, multiplied by y, gives $\frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\frac{1}{2}a-x} =$ the Sub-tangent ST *. Whence (O being ſuppoſed the Center) we have OS $(\frac{1}{2}a-x)$: CS (y) :: CS (y) : ST; which we alſo know from other Principles.

* Art. 48 and 49.

EXAMPLE II.

51. *To draw a Tangent to any given Point* C *of the conical Parabola* ACG.

If the *Latus Rectum* of the Curve be denoted by a, the Ordinate MC by y, and its correſponding Abſciſſa



AM by x; then the known Equation, expreſſing the Relation of x and y, being $ax = y^2$, we have, in this Caſe, $a\dot{x} = 2y\dot{y}$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a}$, and conſequently $\frac{y\dot{x}}{\dot{y}}$ † $= \frac{2y^2}{a} = \frac{2ax}{a} = 2x =$ MF. Therefore the Sub-tangent is juſt the double of its correſponding Abſciſſa AM: Which we likewiſe know from other Principles.

† Art. 48 and 49.

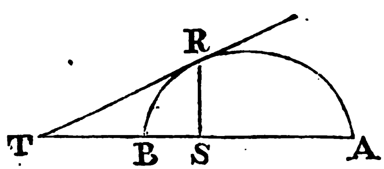
EXAMPLE III.

52. *To draw a Tangent to a Parabola of any kind.*

The general Equation of these sort of Curves being $a^m x^n = y^{m+n}$, we have $na^m x^{n-1}\dot{x} = \overline{m+n} \times y^{m+n-1}\dot{y}$, and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n-1}}{na^m x^{n-1}}$; whence $\frac{y\dot{x}}{\dot{y}} =$ $\frac{\overline{m+n} \times y^{m+n}}{na^m x^{n-1}} = \frac{\overline{m+n} \times a^m x^n}{na^m x^{n-1}}$ (because $y^{m+n} = a^m x^n$) $=$ $\frac{m+n}{n} \times x =$ the true Value of the Subtangent: Which, therefore, is to the Abscissa, in the constant Ratio of $m+n$ to n.

EXAMPLE IV.

53. *To draw a Tangent* RT, *to a given Point* R, *in a given Ellipsis* BRA.



If RS be an Ordinate to the principal Axis AB, and there be put (*as usual*) BS $= x$, RS $= y$, AB $= a$, and the lesser Axis $= b$; we shall, by the Property of the Curve, have $a^2 : b^2 :: ax - x^2$ (BS×AS) : y^2 (RS²), and therefore $b^2 \times \overline{ax - x^2} = a^2y^2$: Whence $b^2 \times \overline{a\dot{x} - 2x\dot{x}} = 2a^2y\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2a^2y}{b^2 \times \overline{a - 2x}}$; and consequently the Sub-tangent ST $\left(\frac{y\dot{x}}{\dot{y}}\right)$* $= \frac{2a^2y^2}{b^2 \times \overline{a-2x}} = \frac{a^2y^2}{b^2 \times \overline{\frac{1}{2}a - x}} = \frac{b^2 \times \overline{ax - x^2}}{b^2 \times \overline{\frac{1}{2}a - x}} =$

* Art. 49.

$ax-$

$\frac{ax-x^2}{\frac{1}{2}a-x}$. Whence the Point T being given, through which the Tangent muſt paſs, the Tangent it ſelf may be drawn.

But if you would derive an Expreſſion for the Subtangent, in any other kind of Ellipſis (beſides the conical) let the Equation $\overline{a-x}|^m \times x^n = \frac{c}{a} \times y^{m+n}$, exhibiting the Nature of all Kinds of Ellipſis, be aſſumed: Then, by taking the Fluxion thereof, you will have $-m\dot{x} \times \overline{a-x}|^{m-1} \times x^n + n\dot{x}x^{n-1} \times \overline{a-x}|^m = \frac{c}{a} \times \overline{m+n} \times y^{m+n-1}\dot{y}$; and therefore $\frac{y\dot{x}}{\dot{y}} =$

$$\frac{\frac{c}{a} \times \overline{m+n} \times y^{m+n}}{-m \times \overline{a-x}|^{m-1} \times x^n + nx^{n-1} \times \overline{a-x}|^m}$$

$= \frac{\overline{m+n} \times \overline{a-x}|^m \times x^n}{-mx^n \times \overline{a-x}|^{m-1} + nx^{n-1} \times \overline{a-x}|^m}$ (becauſe $\frac{c}{a} \times y^{m+n} = \overline{a-x}|^m \times x^n$) $= \frac{\overline{m+n} \times \overline{a-x} \times x}{-mx+n \times \overline{a-x}} =$

$\frac{\overline{m+n} \times \overline{ax-x^2}}{na-\overline{n+m} \times x}$; which is the Sub-tangent required.

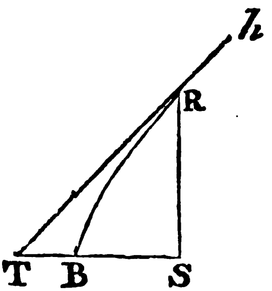
EXAMPLE V.

54. *To draw a Tangent, to any given Point* R, *in a given Hyperbola* BRb.

If a and c be put to denote the two principal Diameters of the Hyperbola, the Equation of the Curve will be $c^2 \times \overline{ax+x^2} = a^2y^2$: From whence we have $c^2 \times$

$\overline{ax+}$

$\overline{a\dot{x}+2x\dot{x}} = 2a^2y\dot{y}$, $\therefore \frac{\dot{x}}{\dot{y}} = \frac{a^2y}{c^2\times\overline{\frac{1}{2}a+x}}$, and consequently $\frac{y\dot{x}}{\dot{y}} = \frac{a^2y^2}{c^2\times\overline{\frac{1}{2}a+x}} = \frac{c^2\times\overline{ax+x^2}}{c^2\times\overline{\frac{1}{2}a+x}} = \frac{ax+x^2}{\frac{1}{2}a+x} = ST$. Whence BT (ST—BS) $= \frac{\frac{1}{2}ax}{\frac{1}{2}a+x}$ is also known; and therefore the Point T being given the Tangent RT may be drawn.



The Manner of drawing Tangents to all Sorts of Hyperbola's, *universally*, will be the same as in the Ellipses, the Equations of the two Kinds of Curves differing in Nothing but their Signs.

EXAMPLE VI.

55. *Let the proposed Curve be that whose Equation is* $ax^2+xy^2+x^3-y^3=0$.

Then we shall have $2ax\dot{x}+y^2\dot{x}+2xy\dot{y}+3x^2\dot{x}-3y^2\dot{y}=0$; therefore $2ax\dot{x}+y^2\dot{x}+3x^2\dot{x} = 3y^2\dot{y}-2xy\dot{y}$, $\frac{\dot{x}}{\dot{y}} = \frac{3y^2-2xy}{2ax+y^2+3x^2}$, and consequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3-2xy^2}{2ax+y^2+3x^2}$*.

* Art. 48 and 49.

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EXAMPLE VII.

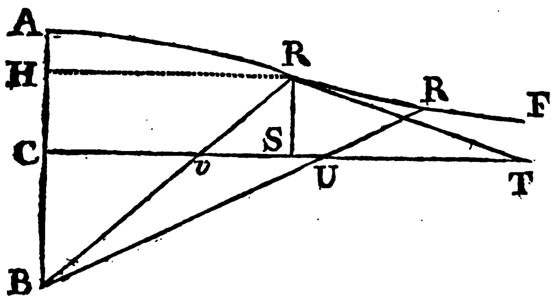
56. *Let the given Curve be the Cissoid of* Diocles, *whose Equation is* $y^2 = \frac{x^3}{a-x}$.

Here we have $2y\dot{y} = \frac{3x^2\dot{x}\times\overline{a-x}+\dot{x}x^3}{\overline{a-x}^2} = \frac{3ax^2\dot{x}-2x^3\dot{x}}{\overline{a-x}^2}$:

Whence $\frac{\dot{x}}{\dot{y}} = \frac{2y\times\overline{a-x}^2}{3ax^2-2x^3}$, and consequently the Subtangent $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2y^2\times\overline{a-x}^2}{3ax^2-2x^3} = \frac{2x^3}{a-x}\times\frac{\overline{a-x}^2}{3ax^2-2x^3} = \frac{2x\times\overline{a-x}}{3a-2x}$.

EXAMPLE VIII.

57. *Let the Conchoid of* Nicomedes *be proposed*; whereof the Nature is such, that, if from a Point B, called



the Pole, any Number of Right-lines, BA, BR, BR, &c. be drawn, the Parts of those Lines CA, vR, UR, &c. intercepted by the Curve and its Axis CT, shall be, all, equal to each other.

In

In this Case (supposing AB and RS perpendicular, and RH parallel, to CT; and putting BC $=a$, Rv (AC) $=b$, CS $=x$, and RS $=y$) we have, *per sim. Triang.* $a+y$ (BH) : x (RH) :: y (RS) : $\frac{xy}{a+y}=$ Sv: But Sv ($\sqrt{Rv^2-RS^2}$) is also $=\sqrt{b^2-y^2}$; therefore $\frac{xy}{a+y}=\sqrt{b^2-y^2}$, or $x^2y^2=\overline{a+y}^2\times\overline{b^2-y^2}$ is the general Equation of the Curve; which, in Fluxions, gives $2x^2y\dot{y}+2y^2x\dot{x}=2\dot{y}\times\overline{a+y}\times\overline{b^2-y^2}-2y\dot{y}\times\overline{a+y}^2=2\dot{y}\times\overline{a+y}\times\overline{b^2-ay-2y^2}$; and therefore $\frac{\dot{x}}{\dot{y}}=\frac{\overline{a+y}\times\overline{b^2-ay-2y^2}-x^2y}{xy^2}$, consequently $\frac{y\dot{x}}{\dot{y}}=\frac{\overline{a+y}\times y\times\overline{b^2-ay-2y^2}-x^2y^2}{y\times xy}=\frac{\overline{a+y}\times y\times\overline{b^2-ay-2y^2}-\overline{a+y}^2\times\overline{b^2-y^2}}{y\times\overline{a+y}\times\sqrt{b^2-y^2}}$ (because $x^2y^2=\overline{a+y}^2\times\overline{b^2-yy}$) $=\frac{b^2y-ayy-2y^3-abb+ayy-bby+y^3}{y\sqrt{bb-yy}}=\frac{-ab^2-y^3}{y\sqrt{bb-yy}}$: Which being a negative Quantity, the Tangent will therefore fall on the contrary Side of the Ordinate from the Vertex; and so, by changing the Signs we shall have $\frac{abb+y^3}{y\sqrt{bb-yy}}$ for the Sub-tangent ST in this Case.

After the Manner of these Examples the Sub-tangent, in Curves whose Abscissas are Right-lines, may be determined: But if the Abscissa, or Line terminating the Ordinates on the lower Part, be another Curve, then the Tangent may be drawn as in the following

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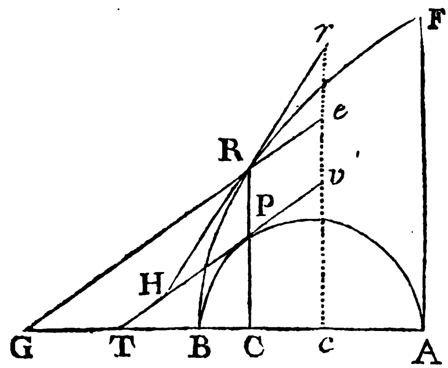
EXAMPLE IX.

58. Let the Curve BRF be a Cycloid; whose Abſciſſa is here ſuppoſed to be the Semicircle BPA, to which let the Tangent PT be drawn (as above). Moreover let *r*RH be a Tangent to the Cycloid, at the cor-



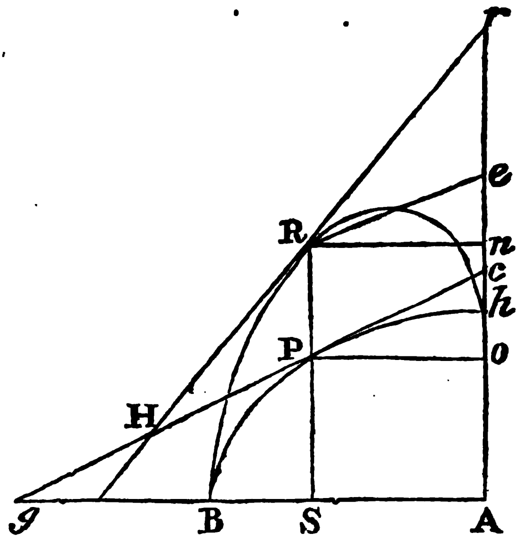
reſponding Point R, and let GR*e* be parallel to TP*v*; putting the Arch (or Abſciſſa) BP$=z$, its Ordinate PR$=y$, AF$=b$, and BPA$=c$: Then, by the Property of the Curve, we ſhall have c (BPA) : b (AF) :: z (BP) : y (PR) : Therefore $y=\frac{bz}{c}$, and $\dot{y}=\frac{b\dot{z}}{c}=re$: But, by ſimilar Triangles, re $(\dot{y})$: Re ($=$ Pv $=\dot{z}$) :: PR (y) : PH $=\frac{y\dot{z}}{\dot{y}}=z$ (becauſe $y=\frac{bz}{c}$). Therefore, if in the Right-line PT, there be taken PH equal to the Arch PB, you will have a Point H, through which the Tangent of the Cycloid muſt paſs.

EXAMPLE X.

59. Let BP*b* be a Curve of any Kind, to which the Method of drawing the Tangent *c*P*g* is known; let

BR*b*

BRb be another Curve of ſuch a Nature, that the Ordinate PR (y) ſhall always be a Mean-proportional be-



tween BS (x) and AS $(a-x)$ ſuppoſing RPS perpendi-
Art. 48 and 49. cular to AB: Put P$o = \dot{x}$, SP $= v$, $oc = \dot{v}$ *, and $er = \dot{y}$: Then, *(as above)* er $(\dot{y})$: Re $(= Pc = \sqrt{\dot{x}^2 + \dot{v}^2})$ :: RP (y) : PH $= \frac{y\sqrt{\dot{x}^2 + \dot{v}^2}}{\dot{y}}$: But, by the Equation of the Curve $y^2 = ax - xx$; whence $2y\dot{y} = a\dot{x} - 2x\dot{x}$, and $\frac{y}{\dot{y}} = \frac{2ax - 2x^2}{a\dot{x} - 2x\dot{x}}$, and therefore PH $= \frac{\overline{2ax - 2x^2} \times \sqrt{\dot{x}^2 + \dot{v}^2}}{a\dot{x} - 2x\dot{x}}$: Which will be expreſſed independent of Fluxions, when the Property of the Curve BPb, or the Relation of x and v is given: Thus, let BPb be the common Parabola, and AB its *Latus Rec-*

tum; then v being $=\sqrt{ax}$, $\dot{v}$ will be $=\dfrac{a\dot{x}}{2\sqrt{ax}}$,

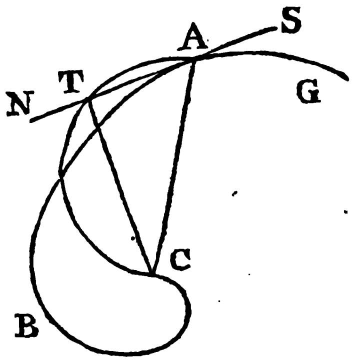
$\dot{x}^2+\dot{v}^2=\dot{x}^2+\dfrac{a\dot{x}\dot{x}}{4x}=\dfrac{\dot{x}\dot{x}\times\overline{4x+a}}{4x}$; and therefore PH

$$\left(\frac{\overline{2ax-2xx}\times\sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x}-2x\dot{x}}\right)=\frac{\overline{a-x}\times\sqrt{4x^2+ax}}{a-2x}.$$

Thus far relates to Curves whose Ordinates are parallel to each other: We come now to Curves of the spiral Kind, whose Ordinates all issue from a Point: Such as the Spiral BAG, whose Ordinates CB, CA, CG, are all referred to the Point C, called the Center of the Spiral.

ILLUSTRATION.

60. Let SAN be a Tangent to the Spiral at any Point A, also let CT be perpendicular thereto, and let the Arch CBA (considered as variable by the Motion of A towards G) be denoted by z, and the Ordinate CA by y.



Then $\dot{z}:\dot{y}::$ AC (y) : AT $=\dfrac{y\dot{y}}{\dot{z}}$.*

* Art. 5 and 35.

Hence, if upon CA, as a Diameter, a Semi-circle be described, and in it, from A, a Right-line AT equal to $\dfrac{y\dot{y}}{\dot{z}}$ be inscribed, that Right-line will be a Tangent to the Spiral at the Point A.

EXAMPLE I.

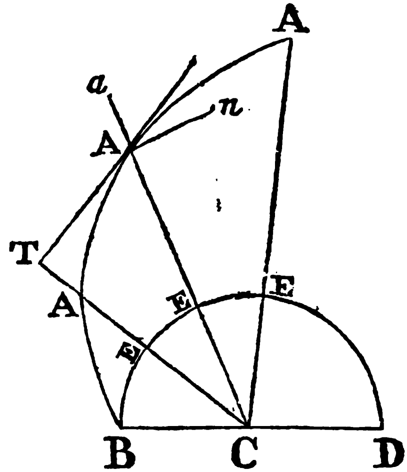
61. Let the Nature of the Curve CBA be such that the Arch CBA may be, always, to its cor-

responding

responding Ordinate CA in a conſtant Ratio; namely as a to b: Then, becauſe $z : y :: a : b$, we have $z = \frac{ay}{b}$, $\dot{z} = \frac{a\dot{y}}{b}$, and conſequently AT $\left(\frac{y\dot{y}}{\dot{z}}\right) = \frac{by}{a} = \frac{b}{a} \times$ AC : Therefore, AC and AT being in a conſtant Ratio, the Angle CAT muſt alſo be invariable. Which is a known Property of the logarithmic Spiral.

EXAMPLE II.

62. Let BAA be the Spiral of *Archimedes*; whoſe Nature is ſuch that the Part EA of the generating Ordinate, intercepted by the Spiral and a Circle BED deſcribed about the ſame Center C, is always in a conſtant Ratio to the correſponding Arch BE of that Circle.



Suppoſe An perpendicular to AC, &c.

Put BC$=c$, CA$=y$, and let the given Ratio of AE to BE, be that of b to c: Then $b : c :: y-c$ (AE) : $\frac{cy-cc}{b}$ = BE: whoſe Fluxion therefore is $= \frac{c\dot{y}}{b}$. Now if if the Right-line CEAa be ſuppoſed to revolve about the Center C, the angular Celerity of the generating Point A, in the perpendicular Direction An, will be to that of E as AC to EC; therefore as the latter of theſe Celerities is expreſſed by $\frac{c\dot{y}}{b}$ *, the former will be expreſſed by $\frac{y}{c} \times \frac{c\dot{y}}{b}$, or $\frac{y\dot{y}}{b}$: Which is to ($\dot{y}$) the Celerity of A, in the Direction Aa, as $\frac{y}{b}$ to Unity, or as y to b. Therefore CT and AT are in the ſame Ratio, (by *Art.* 35) and conſequently AC : CT :: $\sqrt{yy+bb}$: y; and AC : AT :: $\sqrt{yy+bb}$: b; whence CT and AT are given equal to $\frac{y^2}{\sqrt{yy+bb}}$, and $\frac{by}{\sqrt{yy+bb}}$ reſpectively. From either of which (the Tangent AT may be drawn by *Art.* 60. And, in the ſame manner may the Poſition of the Tangent of any other Spiral be determined.

* Art. 5.

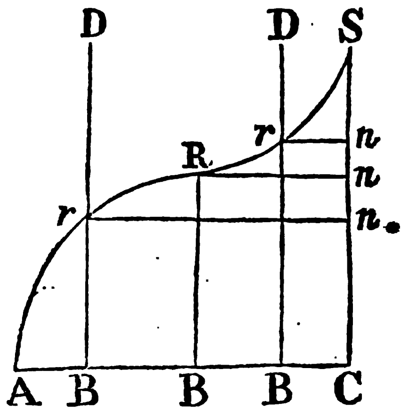
SECTION IV.

Of the Uſe of Fluxions in determining the Points of Retrogreſſion, or contrary Flexure in Curves.

63. WHEN a Curve ARS is, in one Part AR concave, and in the other Part RS convex, towards its Axis AC, the Point R limiting the two Parts is called a Point of Retrogreſſion, or contrary Flexure. The manner of determining which will appear from the following

ILLUSTRATION.

Suppose a Right-line BD to be carried along uniformly, parallel to itself, from A towards C; and let the Point r so move in that Line, at the same time, as to trace out, or describe, the given Curve-line ARS.



Then (*by Art.* 48.) while the Celerity of the Point r, in the Line BD, decreases, the Curve will be concave to its Axis AC; but when it increases, convex to the same: Therefore, as any Quantity is a *Minimum* at the End of its Decrease and the Beginning of its In-
* Art. 22 crease *, it follows that the said Celerity, at the Point of Inflexion R, must be a *Minimum*: Whence, if the
† Art. 5. Fluxion of the Ordinate Br, expressing that Celerity †, be (as usual) denoted by $\dot{y}$; then will $\ddot{y}$ (the Fluxion
‡ Art. 22 of $\dot{y}$) be equal to Nothing in that Circumstance ‡.

So far relates to Curves which are, in the former Part concave, and in the latter convex, to their Axes: But if (on the contrary) the Celerity of r first increases, and then decreases, that Celerity, at the required Point, between the Increase and Decrease, will be a *Maximum*, and therefore its Fluxion (or $\ddot{y}$) is *likewise* equal to
§ Art. 22. Nothing in this Case §.

Furthermore, if CS (perpendicular to AC) be now considered as an Axis, and the Abscissa Sn (or its Complement $Br = y$) be supposed to flow uniformly, (as AB was supposed before); then, by the same Argument, the second Fluxion ($-\ddot{x}$) of the ordinate nr (or (or its Complement AB $=x$) will be equal to Nothing. Hence it is evident that, at the Point of contrary Flexure, the ſecond Fluxion of the Ordinate will become equal to Nothing, if the Abſciſſa be made to flow uniformly; and *vice verſâ.*

EXAMPLE I.

64. Let the Nature of the Curve ARS (*ſee the preceding Figure*) be defined by the Equation $ay = a^{\frac{3}{2}}x^{\frac{1}{2}} + xx$ (the Abſciſſa AB and the Ordinate Br being, as uſual, repreſented by x and y reſpectively). Then $\dot{y}$, expreſſing the Celerity of the Point r, in the Line BD, will be equal to $\frac{\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} + 2x\dot{x}}{a}$: Whoſe Fluxion, or that of $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} + 2x$ (becauſe a and $\dot{x}$ are conſtant) muſt be equal to Nothing *; that is, $-\frac{1}{4}a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x} + 2\dot{x} = 0$: Whence $a^{\frac{3}{2}}x^{-\frac{3}{2}} = 8$, $a^{\frac{3}{2}} = 8x^{\frac{3}{2}}$, $64x^3 = a^3$, and $x = \frac{1}{4}a = $ AB; therefore BR $\left(= \frac{a^{\frac{3}{2}}x^{\frac{1}{2}} + xx}{a}\right) = \frac{9}{16}a$: From which the Poſition of the Point R is given.

* Art. 63.

EXAMPLE II.

65. Let the Nature of the propoſed Curve be defined by the Equation $ayy - aax - x^3 = 0$.

Then, by taking the firſt and ſecond Fluxions thereof (ſuppoſing $\dot{x}$ conſtant) we ſhall alſo have $2ay\dot{y} - aa\dot{x} - 3x^2\dot{x} = 0$, and $2a\dot{y}^2 + 2ay\ddot{y} - 6x\dot{x}\dot{x} = 0$; whereof the latter, when $\ddot{y}$ is $= 0$, becomes $2a\dot{y}^2 - 6x\dot{x}^2 = 0$, and therefore $\dot{y}^2 = \frac{3x\dot{x}^2}{a}$: But, by the former $\dot{y} = \frac{a^2\dot{x} + 3x^2\dot{x}}{2ay}$; whence $\frac{3x\dot{x}^2}{a} = \frac{\overline{a^2\dot{x} + 3x^2\dot{x}}^2}{\overline{2ay}^2}$, and conſequently $12axy^2$

$=\overline{a^2+3x^2}|^2$; but, by the given Equation, $12axy^2 = 12a^2x^2+12x^4$, therefore $12a^2x^2+12x^4 = \overline{a^2+3x^2}|^2$, or $3x^4+6a^2x^2-a^4=0$: Whence x will be found $= a\sqrt{:\sqrt{\tfrac{4}{3}}-1}$.

Otherwise.

Since $ay^2 = a^2x+x^3$, we have $y = \dfrac{\overline{a^2x+x^3}|^{\frac{1}{2}}}{\sqrt{a}}$, and therefore $\dot{y} = \dfrac{\overline{\tfrac{1}{2}a^2\dot{x}+\tfrac{3}{2}x^2\dot{x}} \times \overline{a^2x+x^3}|^{-\frac{1}{2}}}{\sqrt{a}}$: Whose Fluxion, or that of $\overline{a^2+3x^2} \times \overline{a^2x+x^3}|^{-\frac{1}{2}}$ (because $\dot{x}$ is constant) being put $=0$, we get $6x \times \overline{a^2x+x^3}|^{-\frac{1}{2}} + \overline{a^2+3x^2} \times \overline{-\tfrac{1}{2}a^2-\tfrac{3}{2}x^2} \times \overline{a^2x+x^3}|^{-\frac{3}{2}} = 0$, or $6x \times \overline{a^2x+x^3} + \overline{a^2+3x^2} \times -\dfrac{\overline{a^2+3x^2}}{2}$: Whence $3x^4+6a^2x^2-a^4=0$, and $x = a\sqrt{:\sqrt{\tfrac{4}{3}}-1}$, the same as before.

EXAMPLE III.

66. Let the proposed Curve be the Conchoid of *Nicomedes*, whereof the Equation is $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-y^2}$ *, or $x^2 = \dfrac{\overline{a+y}|^2 \times \overline{b^2-y^2}}{y^2}$.

* Art. 57.

Here

Here we have $x\dot{x} = \frac{\overline{\overline{\dot{y}\times\overline{a+y}\times\overline{b^2-y^2}} - y\dot{y}\times\overline{a+y}^2}\times y^2 - y\dot{y}\times\overline{a+y}^2\times\overline{b^2-y^2}}{y^4} = -\frac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y} = \overline{-\frac{a^2b^2}{y^3}-\frac{ab^2}{y^2}-a-y}\times\dot{y}$: Whence, making $\dot{y}$ invariable, we also have $\dot{x}^2+x\ddot{x} = \overline{\frac{3a^2b^2}{y^4}+\frac{2ab^2}{y^3}-1}\times\dot{y}^2$: Which, becaufe $\ddot{x}$ is $=0$ *, will be $\dot{x}^2 = \overline{\frac{3a^2b^2}{y^4}+\frac{2ab^2}{y^3}-1}\times\dot{y}^2 = \frac{3a^2b^2+2ab^2y-y^4}{y^4}\times\dot{y}^2$. But fince, by the former Equation, $x\dot{x} = -\frac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y}$, we likewife get $\dot{x}^2 = \frac{\overline{a+y}^2\times\overline{ab^2+y^3}^2}{x^2y^6}\times\dot{y}^2$, and confequently $\overline{3a^2b^2+2ab^2y-y^4}\times x^2y^2 = \overline{a+y}^2\times\overline{ab^2+y^3}^2$: But by the Equation of the Curve x^2y^2 is $= \overline{a+y}^2\times\overline{b^2-y^2}$; therefore $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{a+y}^2\times\overline{b^2-y^2} = \overline{a+y}^2\times\overline{ab^2+y^3}^2$, and $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{b^2-y^2} = \overline{ab^2+y^3}^2$; whence $y^4+4ay^3+3a^2y^2-2ab^2y-2a^2b^2=0$; which divided by $y+a$, gives $y^3+3ay^2-2ab^2=0$; from whence y may be determined. But if $b=a$, the Equation will become more fimple by dividing again by $y+a$; in which Cafe we get $y^2+2ay-2a^2=0$, and confequently $y=a\sqrt{3}-a$.

* Art. 63.

EXAMPLE IV.

67. Let $a^4y = 180a^3x^2 - 110a^2x^3 + 30ax^4 - 3x^5$.

Then will $a^4\dot{y} = 360a^3x\dot{x} - 330a^2x^2\dot{x} + 120ax^3\dot{x} - 15x^4\dot{x}$;

And $a^4\ddot{y} = 360a^3\dot{x}^2 - 660a^2x\dot{x}^2 + 360ax^2\dot{x}^2 - 60x^3\dot{x}^2$. Therefore, $6a^3 - 11a^2x + 6ax^2 - x^3 = 0$ *:

* Art. 63.

Which being divisible by any one of the three Quantities $a-x$, $2a-x$, or $3a-x$, the Root x must therefore have three Values, a, $2a$, and $3a$, and consequently the Curve, defined by the given Equation, as many Points of contrary Flexure.

But, if you would know whether the Part of the Curve lying between any two adjacent Points, thus found, be convex or concave towards the Axis; see whether the Value of the Expression for the second Fluxion of the Ordinate, between the two corresponding Roots, be positive or negative: For, in the former Case, the Curve is convex, and in the latter concave †, (provided the *whole* Curve lies on the same Side the Axis). Thus, in the Example before us; because the second Fluxion of the Ordinate is always as $6a^3 - 11aax + 6axx - x^3$ $(= \overline{a-x} \times \overline{2a-x} \times \overline{3a-x})$ and it appears that the Value of this Expression, while x is less than the first Root a, will be positive; the Curve, therefore, at the Beginning, will be convex to its Axis: But when x becomes greater than a, the said Expression being negative, the Curve will then be concave, and so continue 'till x is equal to the second Root $2a$; after which the Fluxion again becoming affirmative, the Curve will accordingly be convex till $x = 3a$; beyond which Limit the Curvature continually tends the same Way.

† Art. 5 and 48.

But it will be proper to observe, that there are Cases where the second Fluxion of the Ordinate may become equal to Nothing, without either changing its Value from positive to negative, or the contrary, (similar to those already taken Notice of in *Sect.* II. *p.* 45 and 46.) which Cases always happen when the Equation admits of an even Number of equal Roots: And then the Point found as above is not a Point of Inflexion, because the Curvature on either Side of it tends the same Way.

SECT.

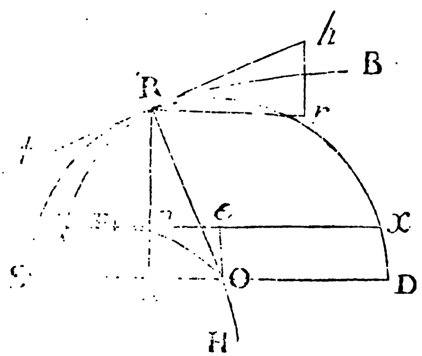
SECTION V.

The Use of Fluxions in determining the Radii of Curvature, and the Evolutes of Curves.

68. A Curve *p*OH is said to be the Evolute of another Curve ARB, when it is of such a Nature, that a Thread ROH, coinciding therewith (or wrapped upon the same) being unwound or disengaged from it, by a Power acting at the End R, shall, by that End (the Thread continuing tight) describe the given Curve ARB.

ILLUSTRATION.

From the Point O, where the Right-line RO (called the Radius of Curvature) touches the Involute *p*OH,



let the Semi-circle SRO be described; which Semi-circle, having the same Radius with the given Curve, at R, will consequently have the same Degree of Curvature. —— But the Curvature in two Curves is the same, when, the Fluxions of their Abscissas being the same, both the First, and Second Fluxions of their

corresponding Ordinates Rn and Rm are respectively equal to each other: For, the First Fluxions being equal, the two Curves will have, at the common Point
* Art. 48. R, one and the same Tangent tRb *: And, if the Second Fluxions be likewise equal, the Curvature, or Deflection from that Tangent, will also be the same in both; because these last express the Increase or Decrease
† Art. 19. of Motion in the Direction of the Ordinate †, upon
‡ Art. 48. which the Curvature intirely depends ‡.

This being premised, let the Abscissa Sm of the Semicircle (considered as variable) be put $=w$, its Ordinate R$m=v$, R$r=\dot{w}$, $rb=\dot{v}$, and R$b=\dot{z}$: Then, Rb be-
‖ Art. 48. ing a Tangent to the Circle at R ‖, the Triangles Rbr and ROm will be equiangular, and therefore $\dot{w}$ (Rr) : $\dot{z}$ (RA) :: v (Rm) : RO $=\frac{v\dot{z}}{\dot{w}}$; which, because the Radius of every Circle is a constant Quantity, must be invariable, and consequently its Fluxion $\frac{\dot{v}\dot{z}+v\ddot{z}}{\dot{w}}=0$: Whence v is found $=\frac{\dot{v}\dot{z}}{-\ddot{z}}=\frac{\dot{z}^2}{-\ddot{v}}$ (because, $\dot{w}$ being constant, and $\dot{w}^2+\dot{v}^2=\dot{z}^2$, we have, in Fluxions, $2\dot{v}\ddot{v}=2\dot{z}\ddot{z}$, and so $\frac{\dot{v}\dot{z}}{-\ddot{z}}=\frac{\dot{z}^2}{-\ddot{v}}$). Therefore since v is $=\frac{\dot{z}^2}{-\ddot{v}}$, we also get SO$=RO\left(\frac{v\dot{z}}{\dot{w}}\right)=\frac{\dot{z}^3}{-\dot{w}\ddot{v}}=\frac{(\dot{w}^2+\dot{v}^2)^{\frac{3}{2}}}{-\dot{w}\ddot{v}}$: Which last is a general Expression for the Radius of any Circle whatever, in Terms of the Fluxions of its Abscissa (w) and Ordinate (v). But, by what is premised above, these Fluxions are respectively equal to those of the Abscissa An (x) and Ordinate Rn (y) of the proposed Curve ARB. Therefore, by writing $\dot{x}$, $\dot{y}$, and $\ddot{y}$, instead of $\dot{w}$, $\dot{v}$, and $\ddot{v}$, we have $\frac{(\dot{y}^2+\dot{x}^2)^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ $\left(=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)$ for the general Value of the Radius of Curvature, RO.

The

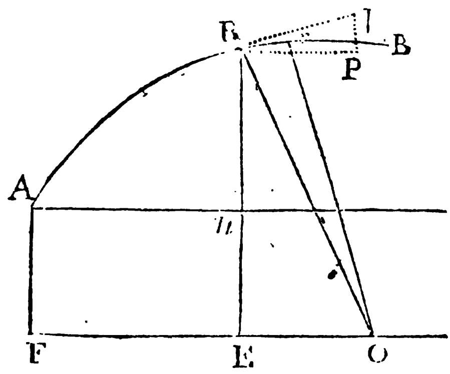
The same otherwise.

If the Radius of the Circle be put $=R$, and every Thing else be supposed as above; then (by the Property of the Circle we shall have v^2 (Rm^2) $=2Rw-w^2$ ($Sm\times Dm$): Whence, in Fluxions (making $\dot{w}$ constant) we get $2v\dot{v}=2R\dot{w}-2w\dot{w}$, and $2\dot{v}^2+2v\ddot{v}=-2\dot{w}^2$: From the last of which Equations v is found $=\frac{\dot{v}^2+\dot{w}^2}{-\ddot{v}}$ $=\frac{\dot{z}^2}{-\ddot{v}}$; and consequently RO $\left(\frac{v\dot{z}}{\dot{w}}\right)=\frac{\dot{z}^3}{-\dot{w}\ddot{v}}=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *the same as before.*

Otherwise without the Circle.

Let RO and rO be two Rays perpendicular to the Curve, indefinitely near to each other; and from their Intersection O, let OF be drawn parallel to An, cutting Rn and AF (parallel to Rn) in E and F.

Therefore, supposing RE$=v$, A$n=x$, R$n=y$, &c. (as before) we shall have, by similar Triangles, as RP



($\dot{x}$) : Pq ($\dot{y}$) :: RE (v) : EO $=\frac{v\dot{y}}{\dot{x}}$; and consequently FO (A$n$+EO) $=x+\frac{v\dot{y}}{\dot{x}}$: Which Value (as well as that

that of AF) continuing the fame whether we regard the Radius RO, or the Radius rO, its Fluxion muft therefore be equal to Nothing; that is, $\dot{x}+\frac{\overline{\dot{v}\dot{y}+v\ddot{y}}\times\dot{x}-v\dot{y}\ddot{x}}{\dot{x}^2}=0$; whence $v=\frac{\dot{x}^3+\dot{x}\dot{y}\dot{y}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$, and confequently RO $\left(\frac{v\dot{z}}{\dot{x}}\right)=\frac{\dot{x}^2\dot{z}+\dot{y}\dot{y}\dot{z}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}=\frac{\dot{x}^2\dot{z}+\dot{y}^2\dot{z}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}=\frac{\dot{z}^3}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$: Which, if $\dot{x}$ is fuppofed conftant, or $\ddot{x}=0$, will become $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *as above.*

But if $\dot{y}$ be fuppofed conftant, it will be $\frac{\dot{z}^3}{\dot{y}\ddot{x}}$. And, if $\dot{z}$ be conftant, it will then be $\frac{\dot{z}\dot{y}}{\ddot{x}}$: For, fince $\dot{x}^2+\dot{y}^2=\dot{z}^2$, by taking the Fluxion thereof, we have $2\dot{x}\ddot{x}+2\dot{y}\ddot{y}=0$; whence $\ddot{y}=-\frac{\dot{x}\ddot{x}}{\dot{y}}$; and therefore RO $\left(=\frac{\dot{z}^3}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}\right)=\frac{\dot{z}^3}{\dot{y}\ddot{x}+\frac{\dot{x}^2\ddot{x}}{\dot{y}}}=\frac{\dot{y}\dot{z}^3}{\overline{\dot{y}^2+\dot{x}^2}\times\ddot{x}}=\frac{\dot{y}\dot{z}}{\ddot{x}}$, *as before.*

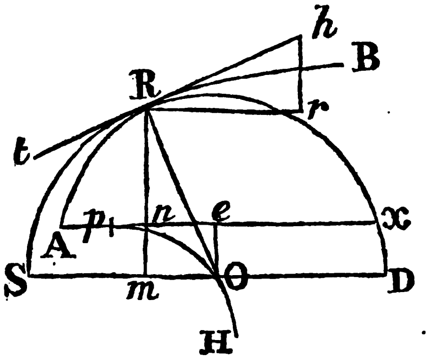
Now from the feveral Values of the Radius of Curvature RO, found above, the correfponding Values of Ae and eO will likewife be given.

Thus, if $\dot{x}$ be made conftant; then, RO being $=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, we fhall have A$e$ (An+Om=An+$\frac{\dot{y}}{\dot{z}}\times$RO) $=x+\frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}$, and eO (Rm—Rn=$\frac{\dot{x}}{\dot{z}}\times$RO—R$n$) $=\frac{\dot{z}^2}{-\ddot{y}}-y$.

But, if $\dot{y}$ be made conftant, then, RO being $=\frac{\dot{z}^3}{\dot{y}\ddot{x}}$, we fhall have AE $=x+\frac{\dot{z}^2}{\ddot{x}}$, and eO $=\frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x}}-y$.

Laftly,

Laſtly, if $\dot{z}$ be ſuppoſed conſtant; then RO being $=\frac{\dot{y}\dot{z}}{\ddot{x}}$, we ſhall have $Ae = x + \frac{\dot{y}^2}{\ddot{x}}$, and $eO = \frac{\dot{x}\dot{y}}{\ddot{x}} - y$.



Which ſeveral Expreſſions will ſerve as ſo many general Theorems for determining the Quantity of Curvature, and the Evolutes of given Curves: But, before we proceed to Examples, it will be proper to obſerve, that the Right-line Ap, denoting the Radius of Curvature at the Vertex A (to be found by making x, or y, $=0$) muſt always be ſubſtracted from RO and Ae, to have the true Length of the Arch pO, and its correſponding Abſciſſa pe.

EXAMPLE I.

69. Let the given Curve ARB be the common Parabola, whoſe Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$: Then will $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$ $= \frac{a^{\frac{1}{2}}\dot{x}}{2x^{\frac{1}{2}}}$, and (making $\dot{x}$ conſtant) $\ddot{y} = -\frac{1}{2} \times \frac{1}{2}a^{\frac{1}{2}}\dot{x}^2x^{-\frac{3}{2}}$ $= \frac{-a^{\frac{1}{2}}\dot{x}^2}{4x^{\frac{3}{2}}}$: Whence $\dot{z}(\sqrt{x+y^2}) = \frac{\dot{x}}{2}\sqrt{\frac{4x+a}{x}}$,

and

and the Radius of Curvature RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a+4x}^{\frac{3}{2}}}{2\sqrt{a}}$: Which at the Vertex A, where $x=0$, will be $=\frac{1}{2}a =$ Ap. Moreover Ae $\left(x+\frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right) = \frac{1}{2}a+3x$, and therefore pe (Ae—Ap) $= 3x$, the Abſciſſa of the Evolute: Likewiſe Oe $\left(\frac{\dot{z}^2}{-\ddot{y}}-y\right) = \frac{4x^{\frac{3}{2}}}{\sqrt{a}}$ the Ordinate of the Evolute. Therefore, $\overline{Oe}^2\times a$ being in a conſtant Ratio to $\overline{pe}^3$, namely as 16 to 27, the Curve is, in this Caſe, the Semi-cubical Parabola ∴ Whoſe Arch pO (RO—Ap) is alſo given $= \frac{\overline{a+4x}^{\frac{3}{2}}}{2\sqrt{a}} - \frac{1}{2}a$.

EXAMPLE II.

70. Let the Curve ARB denote a Parabola of any other Kind: Then, becauſe $y = ax^n$ is an Equation to all Kinds of Parabolas, we have $\dot{y} = nax^{n-1}\dot{x}$ and $\ddot{y} = n\times\overline{n-1}\times ax^{n-2}\dot{x}^2$: Therefore $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}) = \dot{x}\sqrt{1+n^2a^2x^{2n-2}}$, RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{1+n^2a^2x^{2n-2}}^{\frac{3}{2}}}{-n\times\overline{n-1}\times ax^{n-2}}$, A$e$ $\left(x+\frac{\dot{y}\dot{z}^2}{-\ddot{y}}\right) = x - \frac{x+n^2a^2x^{2n-1}}{n-1}$, O$e$ $\left(\frac{\dot{z}^2}{-\ddot{y}}-y\right)$ $= \frac{1+\overline{2n-1}\times na^2x^{2n-2}}{-\overline{n-1}\times nax^{n-2}}$, and A$p = -\frac{n^2a^2 0^{2n-1}}{n-1}$: Which, if $n=\frac{1}{2}$, will become $=\frac{a^2}{2}$; but, if n be greater than $\frac{1}{2}$, it will be $= 0$; and, if n be leſs than $\frac{1}{2}$, it

it will be infinite: Whence it appears, that the Radius of Curvature at the Vertex will be a finite Quantity in Curves whose first (or least) Ordinates are in the Subduplicate Ratio of their Abscissas, and in all other Cases, either Nothing, or Infinite.

EXAMPLE III.

71. Suppose the given Curve to be an Ellipsis; whose Equation (putting a and c for the two principal Diameters) is $a^2y^2 = c^2 \times \overline{ax - x^2}$.

Here, by taking the First and Second Fluxions of the given Equation, we have $2a^2y\dot{y} = c^2\dot{x} \times \overline{a - 2x}$, and $2a^2\dot{y}^2 + 2a^2y\ddot{y} = c^2\dot{x} \times -2\dot{x} = -2c^2\dot{x}^2$; whence $\dot{y} = \frac{c^2\dot{x} \times \overline{a-2x}}{2a^2y}$, and $-\ddot{y} = \frac{a^2\dot{y}^2 + c^2\dot{x}^2}{a^2y}$: Which, by substituting the Values of y and $\dot{y}$, will become $\dot{y} = \frac{c\dot{x} \times \overline{a-2x}}{2a\sqrt{ax-x^2}}$, and $-\ddot{y} = \frac{a^2c^2\dot{x}^2 \times \overline{a-2x}|^2}{4a^2 \times \overline{ax-xx} \times ac\sqrt{ax-x^2}}$

$$+\frac{c\dot{x}^2}{a\sqrt{ax-x^2}} = \frac{c\dot{x}^2}{a} \times \frac{\overline{a-2x}|^2 + 4\times\overline{ax-x^2}}{4\times\overline{ax-x^2}\sqrt{ax-x^2}} = \frac{ca\dot{x}^2}{4\times\overline{ax-x^2}|^{\frac{3}{2}}}:$$

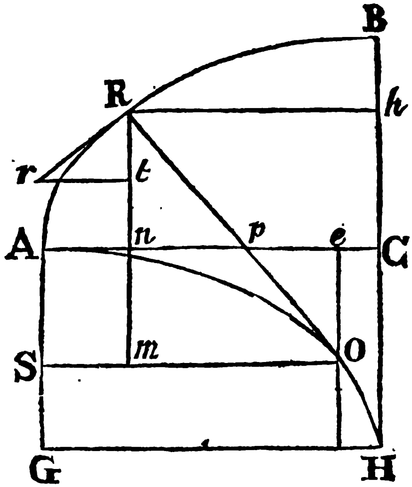
Therefore $\dot{z}\ (\sqrt{\dot{y}^2+\dot{x}^2}) = \sqrt{\frac{c^2\dot{x}^2 \times \overline{a-2x}|^2}{4a^2\times\overline{ax-x^2}} + \dot{x}^2}$

$= \frac{\dot{x}}{2a}\sqrt{\frac{c^2a^2 + \overline{a^2-c^2}\times\overline{4ax-4x^2}}{ax-x^2}}$, and the Radius of Curvature $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a^2c^2 + \overline{a^2-c^2}\times\overline{4ax-4x^2}}|^{\frac{3}{2}}}{2a^4c}$: Which when the Diameters a and c are equal, or the Ellipsis degenerates to a Circle, will be every where equal to $\frac{\overline{a^2c^2}|^{\frac{3}{2}}}{2a^4c}$, or $\frac{1}{2}a$; agreeable to the Definition of a Circle.

EXAMPLE IV.

72. *To find the Radius of Curvature, and the Evolute of the common Cycloid.*

Let ARB be the given Curve, and AOH its Evolute; also let R*b* and OS be parallel to AC, and *e*O and R*m*



perpendicular to AC; and put ARB (=2BC) $= a$, AR$=z$, A$n=x$, and R$n=y$: Then BR$=a-z$, B*b* $= \frac{1}{2}a-y$; and, by the Property of the Curve, a^2 (AB²) : $\overline{a-z}^2$ (BR²) :: $\frac{1}{2}a$ (BC) : $\frac{1}{2}a-y$ (B*b*) whence $y=\frac{2az-z^2}{2a}$; therefore $\dot{y}=\frac{a\dot{z}-z\dot{z}}{a}$, $\dot{z}^2-\dot{y}^2$ $(\dot{x}^2)=\frac{\overline{2az-z^2}\times\dot{z}^2}{a^2}$, and $\dot{x}=\frac{\dot{z}\sqrt{2az-z^2}}{a}$. Whence (making $\dot{z}$ constant) $\ddot{x}=\frac{\dot{z}^2\times\overline{a-z}}{a\sqrt{2az-z^2}}$; from which we get RO, or AO $\left(=\frac{\dot{y}\dot{z}}{\ddot{x}}{}^{*}\right)=\sqrt{2az-z^2}$, and eO, * Art. 68.

or AS $\left(=\frac{\dot{y}\dot{x}}{\ddot{x}}-y\right)=\frac{2az-z^2}{2a}$; which, when $z=a$, or ROH coincides with BH, become AOH (BH) $=a$, and CH (AG) $=\frac{1}{2}a$. Hence, becauſe it appears that, $\overline{AH}^2$ (a^2) : AO^2 $(2az-z^2)$ :: AG $(\frac{1}{2}a)$: AS $\left(\frac{2az-z^2}{2a}\right)$ it follows that the Evolute AOH is alſo a Cycloid equal, and ſimilar, to the Involute ARB.

If the Evolute had been given, or ſuppoſed, a Cycloid, and the Involute required, the Proceſs would have been, more ſimple, as follows,

Let AH (2AG) $=a$, AO $(=$RO$)=z$, AS $=x$, SO$=y$, BR$=v$, B$b$$=w$, R$r$$=\dot{v}$, R$t$$=\dot{w}$, &c. Then it will be †, † Art. 48.

$$\dot{y}:\dot{z}\ (::\ Om:OR)::Rt\ (\dot{w}):Rr=\frac{\dot{w}\dot{z}}{\dot{y}}.$$

$$\dot{z}:\dot{y}::z\ (RO):Om=\frac{z\dot{y}}{\dot{z}},$$

$$\dot{z}:\dot{x}::z\ (RO):Rm=\frac{z\dot{x}}{\dot{z}},$$

Whence we have $\dot{v}=\frac{\dot{w}\dot{z}}{\dot{y}}$, R$n$ (Rm—AS)$=\frac{z\dot{x}}{\dot{z}}-x$, and A$n$ (OS—Om) $=y-\frac{z\dot{y}}{\dot{z}}$; which Expreſſions anſwer to any Curve whatever.

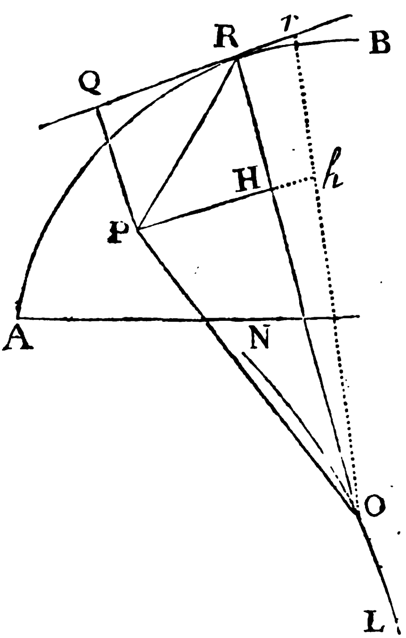
But, in the Caſe above propoſed, AH^2 (a^2) : AO^2 (z^2) :: AG $(\frac{1}{2}a)$: AS (x); therefore $x=\frac{z^2}{2a}$, $\dot{x}=\frac{z\dot{z}}{a}$, and $\dot{y}$ $\left(\sqrt{\dot{z}^2-\dot{x}^2}\right)=\frac{\dot{z}\sqrt{a^2-z^2}}{a}$; and conſequently R$n$ $\left(\frac{z\dot{x}}{\dot{z}}-x\right)=\frac{z^2}{a}-\frac{z^2}{2a}=\frac{z^2}{2a}=\frac{1}{2}a-\dot{w}$ (or CB — Bb):

Whence

Whence alſo $w = \frac{a^2 - z^2}{2a}$, and $\dot{v}\left(\frac{\dot{w}z}{y}\right) = \frac{a\dot{w}}{\sqrt{a^2 - z^2}}$ $= \frac{a\dot{w}}{\sqrt{2aw}}$: Therefore it will be $\dot{v} : \dot{w}$ $(:: a : \sqrt{2aw})$ $:: \sqrt{\frac{1}{2}a} : \sqrt{w}$; that is, as $Rr : Rt :: \sqrt{BC} : \sqrt{Bb}$: Which is a known Property of the Cycloid.

Hitherto regard has been had to Curves where the Ordinates are parallel to each other: But when the Ordinates are all referred to a given Point, as in Spirals, *&c.* other Theorems will become neceſſary; and may be thus derived.

73. Let ARB be the propoſed Curve, P the Point, or Center, to which its Ordinates are referred, NOL the Evolute, and RO the Ray of Curvature at R: Moreover, let PH be perpendicular to RO; and, ſuppoſing the Ordinate PR (y) to become variable by the Motion of the Point R along the Curve, let the Fluxions of AR and PH (p), expreſſing the Celerities of the Points R and H in Directions perpendicular to RO *, be denoted by $\dot{z}$ and $\dot{p}$ reſpectively.



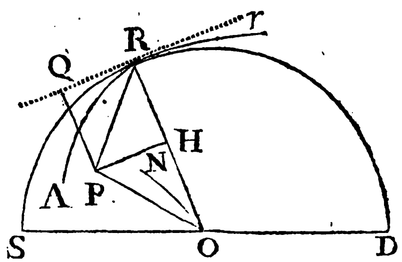
* Art. 5.

Therefore,

Therefore, the Celerities, of any two Points, in a Right-line revolving about a Center, being as the Distances from that Center, it follows that $\dot{p} : \dot{z} :: OH : OR$; whence by Division (putting $RH = v$) we have $\dot{z} - \dot{p} : \dot{z} :: v\ (RH) : RO = \frac{v\dot{z}}{\dot{z}-\dot{p}} = \frac{vp\dot{z}}{p\dot{z}-p\dot{p}}$: But $p\dot{z} = y\dot{y}$ (*by Art.* 60.) and therefore $RO = \frac{vy\dot{y}}{y\dot{y}-p\dot{p}}$; which, because $y^2 - p^2$ is $= v^2$ (and therefore $y\dot{y} - p\dot{p} = v\dot{v}$) will also be $= \frac{vy\dot{y}}{v\dot{v}} = \frac{y\dot{y}}{\dot{v}}$.

The same otherwise.

Let SRD be a Circle described about the Point O, as a Center, and suppose the Distance PR to be variable by the Motion of the Point R along the Arch of the Circle (instead of the Curve): Then, drawing OP, and putting $OR = r$, $PR = y$, &c. as before, we shall get OP^2



$(OR^2 + PR^2 - 2OR \times RH) = r^2 + y^2 - 2rv$; which (as well as r) being a constant Quantity, its Fluxion $2y\dot{y} - 2r\dot{v}$ must be equal to nothing; and therefore $r = \frac{y\dot{y}}{\dot{v}}$, the very same as above. Nor is it of any Consequence whether $\dot{y}$ and $\dot{v}$ be here looked upon as respecting the Circle, or the Curve; since, at R, they must be the same in both Cases; otherwise the Curvature could not be the same *. Now from the Value of RO thus found, which (corrected, when necessary) will also express the Length of the Arch NO of the Evolute ‡, the Ordinate PO and the Tangent OH of the Evolute

* Art. 68.

‡ Art. 68.

may

may be easily deduced. For OH (RO—RH) $=\frac{y\dot{y}}{\dot{v}}$ $-v=\frac{p\dot{p}}{\dot{v}}$, and PO $(=\sqrt{OH^2+PH^2})=\frac{p\sqrt{\dot{p}^2+\dot{v}^2}}{\dot{v}}$; whence the Nature of the Evolute is known.

EXAMPLE I.

74. Let the given Curve AR be the logarithmic Spiral, whose Nature is such, that the Angle PRQ (or RPH) which the Ordinate makes with the Curve is every where the same.

Then (denoting the Sine of that Angle by b, and the Radius of the Tables by a) we have RH $(v)=\frac{by}{a}$, and therefore RO $\left(\frac{y\dot{y}}{\dot{v}}\right)=\frac{ay\dot{y}}{b\dot{y}}=\frac{ay}{b}$; which being to PR (y) in the constant Ratio of a to b, or of PR to RH, the Triangles ROP and RPH must therefore be similar, and so the Angle POH, which the Ordinate PO makes with the Evolute, being every where equal to PRQ, will likewise be invariable. Whence it appears that the Evolute is also a logarithmic Spiral, similar to the Involute; and that a Right-line drawn from the Center, perpendicular to the Ordinate, of any logarithmic Spiral, will pass thro' the Centre of Curvature.

EXAMPLE II.

75. Let the Curve proposed be the Spiral of *Archimedes*; where we have $p=\frac{by}{\sqrt{y^2+b^2}}$, and $v=\frac{y^2}{\sqrt{y^2+b^2}}$ (see *Art.* 62.) Therefore $\dot{v}=2y\dot{y}\times\overline{y^2+b^2}^{-\frac{1}{2}}+yy\times$

$-\frac{1}{2}\times 2y\dot{y}\times\overline{y^2+b^2}^{-\frac{3}{2}} = \frac{2y\dot{y}}{\overline{y^2+b^2}^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}} =$

$\frac{2y\dot{y}\times\overline{y^2+b^2}-y^3\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}} = \frac{y^3\dot{y}+2b^2y\dot{y}}{\overline{y^2+b^2}^{\frac{3}{2}}}$; whence the Radius of

* Curvature $\frac{y\dot{y}}{\dot{v}}$ is here $=\frac{\overline{yy+bb}^{\frac{3}{2}}}{y^2+2b^2}$; which being $=\frac{b}{2}$, Art. 73.

when $y=0$, the Arch of the Evolute †, reckoned from Art. 68.

the Vertex, is therefore $=\frac{\overline{yy+bb}^{\frac{3}{2}}}{y^2+2b^2}-\frac{b}{2}$.

After the very ſame Manner you may proceed in other Caſes: But if the Value of $\dot{v}$ (or $\frac{y\dot{y}}{\dot{v}}$) changes, in any Caſe, from Poſitive to Negative, the Radius of Curvature (RO) after becoming infinite, will fall on the other Side of the Tangent, and the correſponding Point of the Curve, when $\dot{v}=0$, will be a Point of *Contrary-Flexure.* Whence it may be obſerved that the Point of Inflection, in a Curve whoſe Ordinates are referred to a Center, may be found by making the Fluxion of the Perpendicular, drawn from the Center to the Tangent, equal to Nothing, which Caſe is not taken Notice of in the preceding Section.

SECTION VI.

Of the Inverse Method, or the Manner of determining the Fluents of given Fluxions.

76. IN the *Inverse Method*, which teaches the Manner of finding the respective flowing Quantities of given Fluxions, there will be no great Difficulty in conceiving the Reasons, if what is already delivered in *Sect.* 1. *on the direct Method*, has been duly considered: Though the Difficulties that occur in this Part, upon another Account, are indeed vastly superior.

It is an easy Matter, or not impossible at most, to find the Fluxion of any flowing Quantity whatever; but in the *Inverse Method* the Case is quite different: For, as there is no Method for deducing the Fluent from the Fluxion *a priori*, by a direct Investigation, so it is impossible to lay down Rules for any other Forms of Fluxions, than those particular ones which we know, from the direct Method, belong to such and such kinds of flowing Quantities. Thus, for Example, the Fluent of $2x\dot{x}$ is known to be x^2, because it is found in *Art.* 6. and 14. that $2x\dot{x}$ is the Fluxion of x^2: But the Fluent of $y\dot{x}$ is unknown, since no Expression has been discovered that produces $y\dot{x}$ for its Fluxion.

77. Now, as the principal Rule in the *direct Method* is that for the Fluxions of Powers, derived in *Art.* 8. (where it is proved that the Fluxion of x^n is, *universally*, expressed by $nx^{n-1}\dot{x}$); so the most general Rule, that can be given in the *Inverse Method*, must be that arising from the converse thereof; which *shews how to assign the Fluent of any Power of a variable Quantity drawn into the Fluxion of the Root*; and which, expressed in Words, will be as follows.

Divide by the Fluxion of the Root, add Unity to the Exponent of the Power, and divide by the Exponent so increased.

For,

For, dividing the Fluxion $n x^{n-1}\dot{x}$ by $\dot{x}$ (the Fluxion of the Root x) it becomes $n x^{n-1}$; and, adding 1 to the Exponent $(n-1)$ we have $n x^{n}$; which, divided by n, gives x^{n}, the true Fluent of $n x^{n-1}\dot{x}$, by *Art.* 8.

Hence (by the same Rule) the Fluent of $3x^2\dot{x}$ will be $= x^3$;

That of $8x^2\dot{x} = \dfrac{8x^3}{3}$;

That of $2x^5\dot{x} = \dfrac{x^6}{3}$;

That of $y^{\frac{1}{2}}\dot{y} = \frac{2}{3}y^{\frac{3}{2}}$;

That of $ay^{\frac{5}{3}}\dot{y} = \dfrac{3ay^{\frac{8}{3}}}{8}$;

That of $y^{\frac{m}{n}}\dot{y} = \dfrac{y^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \dfrac{ny^{\frac{m+n}{n}}}{m+n}$;

That of $\dfrac{a\dot{x}}{x^n}$, or $a\dot{x}x^{-n}$, $= \dfrac{ax^{1-n}}{1-n}$;

That of $\overline{a+z}^{3} \times \dot{z} = \dfrac{\overline{a+z}^{4}}{4}$;

And that of $\overline{a^m + z^m}^{n} \times z^{m-1}\dot{z} = \dfrac{\overline{a^m + z^m}^{n+1}}{m \times \overline{n+1}}$:

For *here* the Root, or the Quantity under the general Index n, being $a^m + z^m$, and its Fluxion $= m z^{m-1}\dot{z}$ (*Art.* 14.) we shall, by dividing by the last of these Quantities, have $\dfrac{\overline{a^m + z^m}^{n}}{m}$; whence, increasing the

Index by Unity, and dividing by $(n+1)$ the Index so increased, there comes out $\frac{\overline{a^m+z^m}|^{n+1}}{m\times\overline{n+1}}$.

After the very same Manner the Fluents of other Expressions may be deduced, when the Quantity, or Multiplicator, without the Vinculum is either equal, or in a constant Ratio, to the Fluxion of the Quantity under the Vinculum: As in the Expression $\overline{a+cz^n}|^m\times dz^{n-1}\dot{z}$; where the Number of Dimensions of z under the *Vinculum* (or general Index) being equal to those of z without the *Vinculum* $+1$, the Fluent may therefore be had, as in the preceding Examples; and will come out $\frac{\overline{a+cz^n}|^{m+1}\times d}{nc\times\overline{m+1}}$: And, that this (or any other Expression derived in like Manner) is the true Fluent will evidently appear, by supposing x equal to $a+cz^n$ the Quantity under the Vinculum; for then (equal Quantities having equal Fluxions) $\dot{x}$ will be

* Art. 8. $=ncz^{n-1}\dot{z}$ *; and consequently $\overline{a+cz^n}|^m\times dz^{n-1}\dot{z}$ $\left(=x^m\times\frac{d\dot{x}}{nc}\right)=\frac{dx^m\dot{x}}{nc}$; whose Fluent is therefore

‡ Art. 77. $\frac{dx^{m+1}}{nc\times\overline{m+1}}$ ‡ $=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+1}}$, *as before.*

78. In assigning the Fluents of given Fluxions there is another Particular that ought to be attended to, not yet taken notice of; and that is, whether the flowing Quantity, found by the common Rule, above delivered, does not require the Addition or Subtraction of some constant Quantity to render it complete. This indeed

indeed can, only, be known from the Nature of the Problem under Consideration; but that such an Addition or Subtraction may, in some Cases, become necessary is evident from the Subject itself; since a flowing Quantity increased, or decreased, by a constant Quantity, has still the same Fluxion; and therefore the Fluent of that Fluxion is as properly expressed by the whole compound Expression, as by the variable Part of it, alone: Thus, for Instance, the Fluent of $n x^{n-1} \dot{x}$ may be either represented by x^n or by $x^n \pm a$, because (a being constant) the Fluxion of $x^n \pm a$, as well as of x^n, is $n x^{n-1} \dot{x}$.

79. Hence it appears that it is the variable Part of a Fluent only which is assignable by the common Method; the constant Part (when such becomes necessary) being to be ascertained from the particular Nature of the Problem. Now to do this, the best Way is to consider how much the variable Part of the Fluent, first found, differs from the Truth, in that particular Circumstance when the required Quantity which the whole Fluent ought to express, is equal to Nothing; then that Difference, added to, or subtracted from, the said variable Part, as occasion requires, will give the Fluent truly corrected: For, since the Difference of two Quantities flowing with the same Celerity (or having equal Fluxions) is either, Nothing at all, or *constantly* the *same*, the Difference in that Circumstance will likewise be the Difference in all other Circumstances: And therefore being added to the lesser Quantity, or subtracted from the greater, both become equal.

80. To render what is above delivered as familiar as may be, I shall put down a few Examples; in which the variable Quantities represented by x and y are supposed to begin their Existence together, or to be generated, at the same time.

1. Let $\dot{y} = a^2 x\dot{x}$; then the Fluent, found as usual, will be $y = \frac{a^2x^2}{2}$; where taking $y = 0$, $\frac{a^2x^2}{2}$ also vanishes, (because then $x = 0$ by Hypothesis): Therefore the Fluent requires no Correction in this Case.

2. Let $\dot{y} = \overline{a+x}\big|^3 \times \dot{x}$: Here we first have $y = \frac{\overline{a+x}\big|^4}{4}$; but when $y = 0$, then $\frac{\overline{a+x}\big|^4}{4}$ becomes $= \frac{a^4}{4}$ (since x by Hypothesis is then $= 0$:) Therefore $\frac{\overline{a+x}\big|^4}{4}$ *always* exceeds y by $\frac{a^4}{4}$; and so the Fluent properly corrected will *be* $y = \frac{\overline{a+x}\big|^4 - a^4}{4} = a^3 x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$.

But the very same Fluent may be otherwise found, without needing any Correction: For the given Equation $\left(\dot{y} = \overline{a+x}\big|^3 \times \dot{x}\right)$, by expanding $\overline{a+x}\big|^3$, is transformed to $\dot{y} = a^3\dot{x} + 3a^2x\dot{x} + 3ax^2\dot{x} + x^3\dot{x}$; whence $y = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$; the same *as above*.

Hence it appears that the Fluent of an Expression, found according to one Form, may require a very different Correction from the Fluent of the same Fluxion found according to another Form.

3. Let $\dot{y} = \overline{a^2 - x^2}\big|^{\frac{1}{2}} \times x\dot{x}$; then, first, $y = -\frac{\overline{a^2-x^2}\big|^{\frac{3}{2}}}{3}$; where taking $y = 0$, $-\frac{\overline{a^2-x^2}\big|^{\frac{3}{2}}}{3}$ becomes =

$= -\frac{a^3}{3}$; therefore $-\frac{\overline{a^2-x^2}^{\frac{3}{2}}}{3}$ is too little by $\frac{a^3}{3}$; and so the Fluent corrected will be $y = \frac{a^3}{3} - \frac{\overline{a^2-x^2}^{\frac{3}{2}}}{3}$.

4. Let $\dot{y} = \overline{a^m+x^m}^{n} \times x^{m-1}\dot{x}$: Here we first have $y = \frac{\overline{a^m+x^m}^{n+1}}{m\times\overline{n+1}}$; and making $y=0$, the latter Part of the Equation becomes $\frac{\overline{a^m}^{n+1}}{m\times\overline{n+1}} = \frac{a^{mn+m}}{m\times\overline{n+1}}$; whence the Equation, or Fluent, truly corrected is $y = \frac{\overline{a^m+x^m}^{n+1} - a^{mn+m}}{m\times\overline{n+1}}$.

5. Lastly, let $\dot{y} = \overline{a+bx^m+cx^n}^{p} \times \overline{mbx^{m-1}\dot{x}+ncx^{n-1}\dot{x}}$; then, in the first Place, we have $y = \frac{\overline{a+bx^m+cx^n}^{p+1}}{p+1}$; which corrected, as above, becomes $y = \frac{\overline{a+bx^m+cx^n}^{p+1} - a^{p+1}}{p+1}$.

81. Hitherto x and y are both supposed equal to Nothing at the same time; but that will not always be the Case in the Solution of Problems. Thus, for Instance, though the Sine and Tangent of an Arch are both equal to nothing when the Arch itself is equal to Nothing, yet the

the Secant is then equal to the Radius: It will be proper therefore to add an Example or two wherein the Value of y is equal to Nothing, when that of x is equal to any given Quantity a.

Let, then, the Equation $\dot{y}=x^2\dot{x}$ be first proposed; whereof the Fluent (first taken) is $y=\frac{x^3}{3}$; but when $y=0$, then $\frac{x^3}{3}=\frac{a^3}{3}$, by Hypothesis; therefore the Fluent, corrected, is $y=\frac{x^3-a^3}{3}$.

Again, let the proposed Equation be $\dot{y}=-x^n\dot{x}$; then will $y=-\frac{x^{n+1}}{n+1}$; which corrected becomes $y=\frac{a^{n+1}-x^{n+1}}{n+1}$.

Lastly, let $\dot{y}=\overline{c^3+bx^2}^{\frac{1}{2}}\times x\dot{x}$; then, first, $y=\frac{\overline{c^3+bx^2}^{\frac{3}{2}}}{3b}$; and, when $y=0$ and $x=a$, $\frac{\overline{c^3+bx^2}^{\frac{3}{2}}}{3b}$ becomes $=\frac{\overline{c^3+ba^2}^{\frac{3}{2}}}{3b}$: therefore the Fluent corrected is $y=\frac{\overline{c^3+bx^2}^{\frac{3}{2}}-\overline{c^3+ba^2}^{\frac{3}{2}}}{3b}$.

82. All the Examples hitherto given relate to such Fluxions as involve one variable Quantity only in each Term, whose Fluents are assignable from the Converse of the first General Rule, *in Section* I. But, besides these, various other Forms of Fluxions may be proposed, involving two or more variable Quantities, whose Fluents may also be found by Help of the other two General Rules delivered in the same Section.

4 Thus

Thus the Fluent of $y\dot{x}+x\dot{y}$ is expressed by xy *; that * Art. 10.
of $\frac{y\dot{x}-x\dot{y}}{y^2}$ by $\frac{x}{y}$ †; that of $a\dot{x}+x\dot{y}+y\dot{x}$ by $ax+xy$ ‡; † Art. 13. ‡ Art. 10.
and that of $\overline{nx\dot{y}y^{n-1}+y^n\dot{x}-nax^{n-1}\dot{x}}\times\overline{y^nx-ax^n}^{\frac{p}{m}}$ by $\frac{m\times\overline{y^nx-ax^n}^{\frac{p+m}{m}}}{p+m}$: For, dividing (in the last Case) by the Fluxion of the Root $\overline{y^nx-ax^n}$ *, which (by *Art.* * Art. 77.
14 and 15) is $nxy^{n-1}\dot{y}+y^n\dot{x}-nax^{n-1}\dot{x}$, we first have $\overline{y^nx-ax^n}^{\frac{p}{m}}$; whence, adding Unity to the Exponent $\frac{p}{m}$, and dividing by the Exponent so increased, we get

$$\frac{\overline{y^nx-ax^n}^{\frac{p}{m}+1}}{\frac{p}{m}+1}=\frac{m\times\overline{y^nx-ax^n}^{\frac{p+m}{m}}}{p+m}$$

for the true Fluent of the Quantity proposed. But it seldom happens that these Kinds of Fluxions which involve two different variable Quantities in one Term, and yet admit of known, or perfect, Fluents, are to be met with in Practice: I shall therefore take no further Notice of them in this Place (but refer the Reader to the second Part of the Work) my Design here being to insist only upon what is most general and useful in the Subject; which brings me to, further, consider those Forms of Fluxions, involving one variable Quantity only, that frequently occur in the Solution of Problems, whose Fluents may (after proper Transformation) be found, by the Rule already delivered in *Art.* 77.

83. It has been already hinted, that if a Fluxion of the Binomial Kind, as $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$, has the Index $(n-1)$ of the variable Quantity (z) without the *Vinculum* $+1$, equal to (n) the Index of the same Quantity under the *Vinculum*, the Fluent thereof may be then truly found by the forementioned Rule. But the same Observation may be farther extended to *those Cases where the Index without the Vinculum increased by Unity is equal to any Multiple of that under the Vinculum*; as in the Expressions, $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$, $\overline{a+cz^n}|^m \times dz^{3n-1}\dot{z}$, $\overline{a+cz^n}|^m \times dz^{4n-1}\dot{z}$, &c. Whose Fluents are thus determined.

Put $a+cz^n = x$, then will $z^n = \frac{x-a}{c}$, and $nz^{n-1}\dot{z} = \frac{\dot{x}}{c}$ *; and therefore $z^{2n-1}\dot{z} = \frac{x-a}{c} \times \frac{\dot{x}}{nc} = \frac{x\dot{x}-a\dot{x}}{ncc}$; whence by Substitution we get $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z} = \frac{x^m \times d \times \overline{x\dot{x}-a\dot{x}}}{nc^2} = d \times \frac{x^{m+1}\dot{x} - ax^m\dot{x}}{nc^2}$: Whose Fluent (by *Art.* 77) is therefore $= \frac{d}{nc^2} \times \overline{\frac{x^{m+2}}{m+2} - \frac{ax^{m+1}}{m+1}}$; which, by restoring the Value of x, becomes $\frac{d}{nc^2} \times \overline{\frac{\overline{a+cz^n}|^{m+2}}{m+2} - \frac{a \times \overline{a+cz^n}|^{m+1}}{m+1}} =$

* Art. 8.

$d\times$

$\frac{d\times\overline{a+cz^n}^{m+1}}{nc^2}\times\overline{\frac{a+cz^n}{m+2}-\frac{a}{m+1}}=\frac{d\times\overline{a+cz^n}^{m+1}}{nc^2}\times\overline{\frac{cz^n}{m+2}-\frac{a}{\overline{m+2}\times\overline{m+1}}}$; the true Fluent of $\overline{a+cz^n}^m\times dz^{2n-1}\dot{z}$.

Again; for the Fluent of $\overline{a+cz^n}^m\times dz^{3n-1}\dot{z}$, because $z^{n-1}\dot{z}=\frac{\dot{x}}{nc}$, and $z^n=\frac{x-a}{c}$, we have $z^{3n-1}\dot{z}$ $\left(=z^{2n}\times z^{n-1}\dot{z}\right)=\frac{\overline{x-a}^2}{c^2}\times\frac{\dot{x}}{nc}=\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3}$:

Whence, $\overline{a+cz^n}^m$ being $=x^m$, we get $\overline{a+cz^n}^m\times dz^{3n-1}\dot{z}=\frac{dx^m\times\overline{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}}{nc^3}=\frac{d}{nc^3}\times\overline{x^{m+2}\dot{x}-2ax^{m+1}\dot{x}+a^2x^m\dot{x}}$; whose Fluent is therefore $=\frac{d}{nc^3}\times\overline{\frac{x^{m+3}}{m+3}-\frac{2ax^{m+2}}{m+2}+\frac{a^2x^{m+1}}{m+1}}=\frac{dx^{m+1}}{nc^3}\times\overline{\frac{x^2}{m+3}-\frac{2ax}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}^{m+1}}{nc^3}\times\overline{\frac{\overline{a+cz^n}^2}{m+3}-\frac{2aa+2acz^n}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}^{m+1}}{nc^3}\times\overline{\frac{c^2z^{2n}}{m+3}-\frac{2acz^n}{\overline{m+3}\times\overline{m+2}}+\frac{2a^2}{\overline{m+3}\times\overline{m+2}\times\overline{m+1}}}$.

Universally, let r denote any whole positive Number whatever, and let the Fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$ be required; then, by putting $a+cz^n=x$, and proceeding as above, our proposed Fluxion is transformed to $\frac{dx^m\dot{x}}{nc^r}\times\overline{x-a}|^{r-1}$; which, expanding $\overline{x-a}|^{r-1}$ (by the Binomial Theorem) becomes $\frac{d}{nc^r}\times$

$$\overline{x^{m+r-1}\dot{x}-\overline{r-1}\times ax^{m+r-2}\dot{x}+\overline{r-1}\times\frac{r-2}{2}\times a^2x^{m+r-3}\dot{x}}$$

&c. whose Fluent is therefore $=\frac{d}{nc^r}\times\frac{x^{m+r}}{m+r}-$

$$\overline{\frac{\overline{r-1}\times ax^{m+r-1}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{m+r-2}}{2\times\overline{m+r-2}}}\ \text{\&c.} =$$

$$\frac{dx^{m+1}}{nc^r}\times\overline{\frac{x^{r-1}}{m+r}-\frac{\overline{r-1}\times ax^{r-2}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{r-3}}{2\times\overline{m+r-2}}}$$

&c.

Where, r being a whole positive Number, the Multiplicators 1, $r-1$, $\overline{r-1}\times\overline{r-2}$, $\overline{r-1}\times\overline{r-2}\times\overline{r-3}$, &c. will therefore become equal to Nothing, after the r first terms; and so, the Series terminating, the Fluent itself will be truly exhibited in that Number of Terms: Except when $m+r$ is likewise a whole positive Number, less than r; in which Circumstance the Divisors $m+r$, $m+r-1$, $m+r-2$, &c. becoming equal to Nothing, before the Multiplicators, the corresponding Terms of the Series will be infinite. And in that Case the Fluent is said to fail, since Nothing can then be determined from it.

83. Be-

84. Besides the foregoing, there is another Way of deriving the Fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, in Terms of the original flowing Quantity z; which will afford a Theorem more commodious for Practice than that above given: The Method of Investigation is thus.

Let $d\times\overline{a+cz^n}|^{m+1}\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}}$ &c. (where p, v, A, B, C, &c. denote unknown, but determinate, Quantities) be assumed for the Fluent sought: Then by taking the Fluxion of the Quantity so assumed we shall have

$dcn\times\overline{m+1}\times z^{n-1}\dot{z}\times\overline{a+cz^n}|^m\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}\ \&c.}+d\times\overline{a+cz^n}|^{m+1}\times\overline{pAz^{p-1}\dot{z}+\overline{p-v}\times Bz^{p-v-1}\dot{z}+\overline{p-2v}\times Cz^{p-2v-1}\dot{z}\ \&c.}$ which being put* equal to the given Fluxion, $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$, and the whole Equation divided by $\overline{a+cz^n}|^m\times dz^{-1}\dot{z}$, there comes out

*Art. 8. 10.

$$\left.\begin{array}{l}+cn\times\overline{m+1}\times z^n\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}+Dz^{p-3v}\ \&c.}\\ +\overline{a+cz^n}\times\overline{pAz^p+\overline{p-v}\times Bz^{p-v}+\overline{p-2v}\times Cz^{p-2v}\ \&c.}\end{array}\right\}=z^{rn}$$

Whence, by collecting the Coefficients of the like Powers of z, we have

$$\left.\begin{array}{l}\left.\begin{array}{l}n\times\overline{m+1}\\ +p\end{array}\right\}\times cAz^{p+n}+\left.\begin{array}{l}n\times\overline{m+1}\\ +p-v\end{array}\right\}\times cBz^{p+n-v}+\left.\begin{array}{l}n\times\overline{m+1}\\ +p-2v\end{array}\right\}\times cCz^{p+n-2v}\ \&c.\\ -z^{rn}\quad+\quad paAz^p\quad+\overline{p-v}\times aBz^{p-v}\ \&c.\end{array}\right\}=0$$

Where, comparing $p+n$ and rn, the two greatest Exponents of z, we find $p=rn-n=\overline{r-1}\times n$; and by comparing the two next inferior Exponents $p+n-v$, and p, we likewise

likewise get $v=n$; which Values being substituted above, our Equation is reduced to

$$\left.\begin{array}{l}\overline{m+r}\times ncAz^{rn}+\overline{m+r-1}\times ncBz^{rn-n}+\overline{m+r-2}\times ncCz^{rn-2n}\ \&c.\\ -z^{rn}+\quad \overline{r-1}\times naAz^{rn-n}+\overline{r-2}\times naBz^{rn-2n}\quad \&c.\end{array}\right\}=0$$

Where, putting $m+r=s$, and comparing the Coefficients of the homologous Terms *, we have $A=\frac{1}{snc}$, $B=-\frac{\overline{r-1}\times aA}{\overline{s-1}\times c}=-\frac{\overline{r-1}\times a}{s\times\overline{s-1}\times nc^2}$, $C=-\frac{\overline{r-2}\times aB}{\overline{s-2}\times c}=\frac{\overline{r-1}\times\overline{r-2}\times a^2}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}$, $D=-\frac{\overline{r-3}\times aC}{\overline{s-3}\times c}=-\frac{\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times a^3}{s\times\overline{s-1}\times\overline{s-2}\times\overline{s-3}\times nc^4}$, &c. &c.

which Values, with those of p and v, being substituted in the assumed Fluent, it becomes $d\times\overline{a+cz^n}^{\frac{m+1}{n}}\times$ $\overline{\frac{z^{rn-n}}{snc}-\frac{\overline{r-1}\times az^{rn-2n}}{s\times\overline{s-1}\times nc^2}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}}$ &c. $=\frac{d\times\overline{a+cz^n}^{\frac{m+1}{n}}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+}$ $\overline{\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}}$ &c. the true Fluent of $\overline{a+cz^n}^{m+1}\times dz^{rn-1}\dot z$, which was to be determined: Which Fluent therefore, when r is a whole positive Number, will always terminate in as many Terms as are expressed by that Number; except in that particular Case, specified in the last Article. Thus, if $r=2$, or the

* *Vid. p.* 181 *of my Treatise of Algebra.*

the given Fluxion be $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$; then, s $(m+r)$ being $= m+2$, the Fluent itself will become

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+2}} \times \overline{\frac{z^n}{1} - \frac{a}{\overline{m+1}\times c}} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times \overline{\frac{cz^n}{m+2} - \frac{a}{\overline{m+2}\times\overline{m+1}}}$$

; which is exactly the same with the first of those found in *Art.* 83. by a different Method.

The like Agreement will likewise be found, when r is $= 3$: But when r, either denotes a broken, or a negative, Number, the Series for the Fluent will then run on to Infinity; because no one of the Multiplicators $r-1$, $r-2$, $r-3$, $r-4$, *&c.* can in that Case be equal to Nothing.

85. The foregoing Fluent, it may be observed, was found by assuming $d\times\overline{a+cz^n}|^{m+1} \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v}}$ *&c.* and comparing the two greatest Exponents, of the Equation thence resulting: But if, instead of $Az^p + Bz^{p-v} + Cz^{p-2v}$ *&c.* an ascending Series, as $Az^p + Bz^{p+v} + Cz^{p+2v}$ *&c.* (where the Exponents of z continually increase) be taken, and the two least Indices of z in the Equation (in like Manner resulting) be compared together, the same Fluent will be had according to a different Form, which will be of good Use in many Cases, when the foregoing fails, or runs out into an Infinite Series.

Thus, if $p+v$, $p+2v$, *&c.* be wrote in the Room of $p-v$, $p-2v$, *&c.* respectively, in the first Equation of the last Article, it will appear that

$$\left.\begin{aligned}&+cn\times\overline{m+1}\times z^n\times\overline{Az^p+Bz^{p+v}+Cz^{p+2v}\ \&c.}\\&+\overline{a+cz^n}\times\overline{pAz^p+\overline{p+v}\times Bz^{p+v}+\overline{p+2v}\times Cz^{p+2v}\ \&c.}\end{aligned}\right\}=z^{rn}$$

Which Equation may be reduced to

$$\left.\begin{aligned}&paAz^p+\overline{p+v}\times aBz^{p+v}+\overline{p+2v}\times aCz^{p+2v}\ \&c.\\&-z^{rn}+\left.\begin{matrix}n\times\overline{m+1}\\+p\end{matrix}\right\}\times cAz^{p+n}+\left.\begin{matrix}n\times\overline{m+1}\\+p+v\end{matrix}\right\}\times cBz^{p+n+v}\ \&c.\end{aligned}\right\}=0$$

Where, by comparing the two least Exponents, &c. p will be found $=rn$, $v=n$; $A=\frac{1}{pa}=\frac{1}{rna}$; $B=$

$$-\frac{\overline{p+n\times\overline{m+1}}\times cA}{\overline{p+v}\times a}=-\frac{\overline{r+m+1}\times ncA}{\overline{r+1}\times na}=-\frac{\overline{r+m+1}\times c}{r\times\overline{r+1}\times na^2};\ C=-\frac{\overline{p+v+n\times\overline{m+1}}\times cB}{\overline{p+2v}\times a}=-$$

$$\frac{\overline{r+m+2}\times ncB}{\overline{r+2}\times na}=\frac{\overline{r+m+1}\times\overline{r+m+2}\times c^2}{r\times\overline{r+1}\times\overline{r+2}\times na^3}\ \&c.\ \&c.$$

Therefore, denoting $r+m$ by s (as above) the Fluent of $\overline{a+cz^n}\big|^m\times dz^{rn-1}\dot{z}$, will (*also*) be truly represented by

$$d\times\overline{a+cz^n}\big|^{m+1}\times\overline{\frac{z^{rn}}{rna}-\frac{\overline{s+1}\times cz^{rn+n}}{r\times\overline{r+1}\times na^2}+\frac{\overline{s+1}\times\overline{s+2}\times c^2z^{rn+2n}}{r\times\overline{r+1}\times\overline{r+2}\times na^3}\ \&c.}$$

or its Equal

$$\frac{\overline{a+cz^n}\big|^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^n}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^2z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^2}\ \&c.}$$

Which Series will terminate when s (or $r+m$) is a whole negative Number; and therefore in all such Cases the

the Fluent is exactly determined; provided r be not also a negative Integer less than s; for in this particular Circumstance the Fluent fails, the Divisor first becoming equal to Nothing. *Vid. Art.* 83.

The Use of the two foregoing general Expressions, for the Fluent of $\overline{a+cz^n}\,^m \times dz^{rn-1}\dot{z}$, will appear from the following Examples.

EXAMPLE I.

86. *Let it be required to find the Fluent of* $\dfrac{bx\dot{x}}{\overline{a+x}\,^{\frac{1}{2}}}$, or

$$\overline{a+x}\,^{-\frac{1}{2}} \times bx\dot{x}.$$

By comparing the Fluxion here proposed with $\overline{a+cz^n}\,^m \times dz^{rn-1}\dot{z}$, we have $a=a$, $c=1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $d=b$, $rn-1$ (or $r-1$) $=1$; whence $r=2$, and s $(r+m)$ $=\frac{3}{2}$; whereof the former being a whole positive Number, let these Values be therefore substituted in

$$\left(\frac{d\times\overline{a+cz^n}\,^{m+1}}{snc} \times \frac{z^{rn-n}}{1} - \frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c} + \frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}, \&c.\right)$$

the first of the two general Expressions for the Fluent, and it will become $\dfrac{b\times\overline{a+x}\,^{\frac{1}{2}}}{\frac{1}{2}} \times x - \dfrac{a}{\frac{1}{2}} = \dfrac{b\times\overline{a+x}\,^{\frac{1}{2}}\times\overline{2x-4a}}{3}$, the Quantity sought in this Case.

EXAMPLE II.

87. *Let the Fluxion proposed be* $\frac{b\dot{x}x^{3n-1}}{\sqrt{a+fx^n}}$, or

$$\overline{a+fx^n}|^{-\frac{1}{2}} \times bx^{3n-1}\dot{x}.$$

Here, by proceeding as above, we have $a=a$, $c=f$, $z=x$, $n=n$, $m=-\frac{1}{2}$, $d=b$, $r=3$, and s $(r+m)=\frac{5}{2}$: Whence, by substituting these several Values in the same general Expression, we get $\frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{\frac{5}{2}nf}\times$

$$\overline{x^{2n}-\frac{2ax^n}{\frac{3}{2}f}+\frac{2a^2}{\frac{3}{2}\times\frac{1}{2}f^2}} = \frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{nf^3}\times \frac{6f^2x^{2n}-8afx^n+16a^2}{15}.$$

EXAMPLE III.

88. *Wherein the Quantity proposed is* $\frac{\dot{y}\sqrt{g^2+y^2}}{y^6}$, or

$$\overline{g^2+y^2}|^{\frac{1}{2}}+y^{-6}\dot{y}.$$

Here we have $a=g^2$, $c=1$, $z=y$, $n=2$, $m=\frac{1}{2}$, $d=1$, $rn-1$ (or $2r-1$) $=-6$; whence r $\left(=\frac{-6+1}{2}\right)$ $=-\frac{5}{2}$, and s $(r+m)=-2$; whereof the latter being a whole Negative Number, let the several Values here exhibited be therefore substituted in

$\left(\frac{\overline{a+cz^n}|^{m+1} \times dz^{rn}}{rna} \times \overline{1 - \frac{s+1 \times cz^n}{r+1 \times a} + \frac{s+1 \times s+2 \times c^2 z^{2n}}{r+1 \times r+2 \times a^2}}\right.$ &c.) the latter of the two general Expreffions above derived, and it will become $\frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times y^{-5}}{-5g^2} \times \overline{1 - \frac{-1 \times y^2}{-\frac{3}{2} \times g^2}} = \frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times \overline{2y^2 - 3g^2}}{15g^4 y^5}$; the true Fluent required.

EXAMPLE IV.

89. *Laftly, let the given Fluxion be* $\overline{a-fz^n}|^{\frac{1}{2}} \times z^{-\frac{7}{2}n-1}\dot{z}$.

Then, a being $=a$, $c=-f$, $m=\frac{1}{2}$, $d=1$, $r=-\frac{7}{2}$, and the reft as in the general Fluxion $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$; we fhall, by fubftituting in the fecond Form (becaufe s is here equal to (-3) a whole negative Number) have $\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times z^{-\frac{7}{2}n}}{-\frac{7}{2}na} \times \overline{1 - \frac{-2 \times -fz^n}{-\frac{5}{2}a} - \frac{-2 \times -1 \times f^2 z^{2n}}{-\frac{5}{2} \times -\frac{3}{2}a^2}} = \frac{\overline{a-fz^n}|^{\frac{3}{2}}}{-\frac{7}{2}naz^{\frac{7}{2}n}} \times \overline{1 + \frac{4fz^n}{5a} + \frac{8f^2z^{2n}}{15a^2}}$

$$= -\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times \overline{30a^2 + 24afz^n + 16f^2z^{2n}}}{105na^3 z^{\frac{7}{2}n}}.$$

90. Having infifted largely on the Manner of finding fuch Fluents as can be truly exhibited in Algebraic Terms; it remains now to fay fomething with regard

to those other Forms of Expressions, involving one variable Quantity only, which, *yet*, are so affected by compound Divisors and radical Quantities, that their Fluents cannot be *accurately* determined by any Method whatsoever; of which there are innumerable Kinds: But there is one general Method whereby the Fluents of such Expressions are approximated, to any assigned Degree of Exactness; namely, the Method of *Infinite Series*; which it will, therefore, be necessary to explain; so far as relates to the Manner of expounding the Value of any compound Fraction, or surd Quantity, by Help of such a Series.

EXAMPLE I.

91. *Let, then, the Fraction* $\frac{ax}{a-x}$ *be, first, given; to be converted into an Infinite Series.*

Divide the Numerator ax by the Denominator $a-x$, as is taught in Compound Division of common Algebra; then the Operation will stand as follows;

$$a-x)\,ax\quad\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\text{\&c.}\right.$$

$$\begin{array}{l}
\underline{ax-xx} \\
+xx \\
\underline{+xx-\frac{x^3}{a}} \\
\qquad +\frac{x^3}{a} \\
\qquad \underline{+\frac{x^3}{a}-\frac{x^4}{a^2}} \\
\qquad\qquad +\frac{x^4}{a^2} \\
\qquad\qquad\quad \text{\&c.}
\end{array}$$

Where

Where the Quotient, or Series $x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} + \frac{x^6}{a^5}$ &c. infinitely continued, is taken to expound the Value of the proposed Fraction $\frac{ax}{a-x}$.

92. But, though the Series thus arising ought to be carry'd on to an Infinity of Terms, to have the true Value of the Quantity first proposed; or, though the Quotient, continued to ever so great a Number of Terms, will be *still* something defective of the Truth; yet, if the Value of the Quantity (x) in the Numerator be but small in Comparison of the Quantity (a) in the Denominator, the Remainder, after a few Terms in the Quotient, will become so exceeding small, as to be neglected without any considerable Error; and then the Value of the *Whole*, or of the Quantity first proposed, will be, very nearly, exhibited, by taking a small Number of the leading Terms only.

Thus, for Instance, let the Value of a be expounded by 10, and that of x by Unity; then the Remainder $\left(\frac{x^3}{a}\right)$ after the two first Terms of the Quotient, being $= \frac{1}{10}$, this Value, divided by the given Divisor ($a-x=$) 9, will therefore give $\frac{1}{90} = 0{,}01111111$,&c. for the Defect, by taking the two first Terms only: But, if the three first Terms be taken, the Defect will be *still* less considerable; amounting to no more than $\frac{1}{900}$, or 0,00111111 &c.

This may likewise be made to appear, without any regard to the Remainder, by collecting into one Sum, the Values of all the Terms to be taken: For, if only the first two $\left(x + \frac{x^2}{a}\right)$ be proposed, their Sum will be

$=1,1$; which, deducted from the true Value of the given Fraction $\frac{ax}{a-x}$ $\left(=\frac{10}{9}\right) = 1,1111111$ &c. the Difference will come out 0,01, *the very same as before.*

Thus, also, by collecting the Sum of the three, four and five, &c. first Terms of the Series, you will have 1,11; 1,111; and 1,1111 &c. which, being successively deducted from 1,111111111 &c. (as above) there will remain 0,001111 &c. 0,0001111 &c. 0,00001111 &c. for the Errors or Defects in those Cases respectively.

93. From what has been said in the preceding Article it appears, that Infinite Serieses, in Algebra (according to a common Observation) are similar to, or correspond with, Decimal Fractions in common Arithmetick: For, as a Decimal Fraction may be carry'd on to any proposed Number of Places, however great, and yet never amount to a Quantity, which but a very little exceeds the Value of the three or four first Places; so a Series may be infinite with regard to the Number of its Terms, and yet a few of the leading Terms only, may be sufficient to express the Value of the *Whole*, very nearly: Provided, always, that the Series has a sufficient Rate of Convergency, or that its Terms decrease in a pretty large Proportion: For, otherwise, *even*, a great Number of Terms may be used to little Purpose: Thus, in the foregoing Series, $x + \frac{x^2}{a} + \frac{x^3}{a^2}$ &c. if x be taken $= a$, no Number of Terms will be sufficient to exhibit the Value of the corresponding Fraction $\frac{ax}{a-x}$, it being infinite in that Circumstance.

94. Having endeavoured to shew, that the true Value of an infinite Series may be nearly obtained by adding together a few of the first Terms only, I shall now proceed to give other Examples of the Manner of con-

converting fractional, and furd, Quantities into fuch Kinds of Seriefes, in order to the Approximation of the Fluents of Expreffions affected by them.

EXAMPLE II.

Let the Quantity propofed be the Fraction $\frac{c^2}{c^2+2cy+y^2}$;

then, by proceeding as in the firft Example, you will have

$$c^2+2cy+y^2)\,c^2\,\ldots\ldots\,(1-\frac{2y}{c}+\frac{3y^2}{c^2}-\frac{4y^3}{c^3}\text{ \&c.}$$

$$\frac{c^2+2cy+y^2}{-2cy-y^2}$$

$$\frac{-2cy-4y^2-\frac{2y^3}{c}}{+3y^2+\frac{2y^3}{c}\text{ \&c.}}$$

Where, from a few of the firft Terms of the Quotient, the Law of Continuation is manifeft; the Numerators being in Arithmetical Progreffion; and the Signs, + and —, alternately.

EXAMPLE III.

95. *Let the Quantity given be* $\frac{1+x^2-2x^4}{1-x-x^2}$.

Then the Quotient will be $1+x+3x^2+4x^3+5x^4+9x^5+14x^6$ &c. where the *Law of Continuation* is manifeft; being fuch that the Coefficient of each fucceeding Term is equal to the Sum of thofe of the two Terms immediately preceding it.

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EXAMPLE IV.

96. *Let the Radical Quantity* $\sqrt{a^2+x^2}$ *be proposed.*

Here, according to the common Method of extracting the Square Root, the Process will stand as follows:

$$2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\,aa+xx\,\Big(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}\ \&c.$$

$$aa$$

$$+xx$$

$$+xx+\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\ \&c.\ \&c.$$

97. The Law of Continuation in Serieses, thus arising, from radical Quantities, is not easily discovered: But, if you would carry on the Series to any proposed Number of Terms, the Work will be a good deal shortned, by dividing the Remainder by the Divisor, when half that Number of Terms is found (as in common Division) and observing, at the same time, to neglect all such Terms whose Indices would exceed the greatest, or the greatest Plus the common Difference, in the said Remainder, according as the whole Number of Terms proposed to be found is odd, or even.

Thus, if it were proposed to continue the foregoing Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$ to 6 Terms, then the Divisor

(or

(or double Quotient) being $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, and the Remainder $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ (as appears from the laſt Article) the reſt of the Operation will ſtand thus:

$$2a + \frac{x^2}{a} - \frac{x^4}{4a^3} \Big) \frac{x^6}{8a^4} - \frac{x^8}{64a^6} + 0 \Big(\frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$$

$$\frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8}$$

$$- \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8}$$

$$- \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8}$$

$$+ \frac{7x^{10}}{128a^8}$$

Which three Terms thus found being added to thoſe found above, we have $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, for the 6 firſt Terms of an infinite Series exhibiting the Value of $\sqrt{a^2+x^2}$.

98. Another Way of reſolving any radical Quantity, is to aſſume a Series (with unknown Coefficients) for the Value thereof; and then the Series ſo aſſumed being raiſed to the ſecond, third, or fourth Power, *&c.* according as the Root to be extracted is a ſquare, cubic, or biquadratic one, *&c.* an Equation will be obtained (free from Surds) from whence, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Series ſought, will be determined; as in

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EXAMPLE V.

Where it is proposed to extract the Square Root of $a^{2n}+x^{2n}$ in an Infinite Series.

In which Case, assuming $A+Bx^{2n}+Cx^{4n}+Dx^{6n}+$ $+Ex^{8n}$ *&c.* for the required Series, and taking the Square thereof, we have

$$\left.\begin{array}{r} A^2+2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\ \&c. \\ +B^2x^{4n}\ \ +2BCx^{6n}+2BDx^{8n}\ \&c. \\ +C^2x^{8n}\ \&c. \end{array}\right\} = a^{2n}+x^{2n}$$

and consequently

$$\left.\begin{array}{r} A^2\ \ +2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\ \&c. \\ -a^{2n}-\ \ x^{2n}+\ \ B^2x^{4n}+2BCx^{6n}+2BDx^{8n}\ \&c. \\ +\ \ C^2x^{8n}\ \&c. \end{array}\right\} = 0$$

Therefore $A^2-a^{2n}=0$, $2AB-1=0$, $2AC+B^2=0$, $2AD+2BC=0$, $2AE+2BD+C^2=0$, * *&c.* From which we get $A=a^n$; $B\ (=\frac{1}{2A}) = \frac{1}{2a^n}$; $C\ (= -\frac{B^2}{2A}) = -\frac{1}{8a^{3n}}$; $D\ (= -\frac{BC}{A}) = \frac{1}{16a^{5n}}$; $E\ (= -\frac{2BD+C^2}{2A}) = -\frac{5}{128a^{7n}}$ *&c.* whence we have $A+Bx^{2n}+Cx^{4n}+Dx^{6n}$ *&c.* $(=\sqrt{a^{2n}+x^{2n}}) = a^n$

* *Vid. p.* 181 *of my Treatise of Algebra.*

+

$+\frac{x^{2n}}{2a^n}-\frac{x^{4n}}{8a^{3n}}+\frac{x^{6n}}{16a^{5n}}-\frac{5x^{8n}}{128a^{7n}}$ &c. Which Series, if n be expounded by Unity, will become $a+\frac{x^2}{a^2}-\frac{x^4}{8a^3}$ &c. the very same with that in the preceding Article found by the common Method.

EXAMPLE VI.

99. *Let it be required to resolve* $\overline{a+bx^n}|^{\frac{1}{3}}$ *into an Infinite Series.*

Here, by assuming $A+Bx^n+Cx^{2n}+Dx^{3n}$ &c. and cubing the same, &c. we have

$$\left.\begin{array}{l} A^3+3A^2Bx^n+3A^2Cx^{2n}+3A^2Dx^{3n}+\text{ \&c.} \\ -a-bx^n \quad +3AB^2x^{2n}+6ABCx^{3n}+\text{ \&c.} \\ \qquad\qquad\qquad + \quad B^3x^{3n}+\text{ \&c.} \end{array}\right\}=0$$

Therefore $A=a^{\frac{1}{3}}$; $B\left(=\frac{b}{3A^2}\right)=\frac{b}{3a^{\frac{2}{3}}}$; $C\left(=-\frac{B^2}{A}\right)=-\frac{b^2}{9a^{\frac{5}{3}}}$; $D\left(=-\frac{6ABC+B^3}{3A^2}\right)=\frac{5b^3}{81a^{\frac{8}{3}}}$ &c.

and consequently, $\overline{a+bx^n}|^{\frac{1}{3}}$ $(=A+Bx^n+Cx^{2n}+$&c.$)$ $=a^{\frac{1}{3}}+\frac{bx^n}{3a^{\frac{2}{3}}}-\frac{b^2x^{2n}}{9a^{\frac{5}{3}}}+\frac{5b^3x^{3n}}{81a^{\frac{8}{3}}}+$ &c.

And, in the same Manner, may the Root of any other Quantity be extracted: But as the celebrated Binomial Theorem, discovered by the illustrious Sir *Isaac Newton*, is vastly more easy and expeditious, in raising Powers and extracting Roots than that, or any other, Method, I shall now explain the Uses thereof; but, first

first of all, it may not be amiss to shew how the Theorem itself, from the Principles of Fluxions, may be derived.

Let, then, $1+y$ be a Binomial whose first Term is Unity, and its second Term any proposed Quantity y; and let the Quantity to be expanded or thrown into a Series be $\overline{1+y}^v$; where the Exponent v is supposed to denote any Number whatever, whole or broken, positive or negative.

Now it is evident that the first Term of the required Series must be Unity; because when y is $=0$, the other Terms all vanish; and, in that Case, $\overline{1+y}^v$ is equal to Unity. Let, therefore, $1+Ay^m+By^n+Cy^p+Dy^q$ &c. be assumed to express the true Value of the said Series, or, which is the same, let

$\overline{1+y}^v=1+Ay^m+By^n+Cy^p+Dy^q$ &c. where A, B, C, D, &c m, n, p, q, &c: denote unknown, but determinate Quantities:

Then, by taking the Fluxion of the whole Equation, (supposing y variable) we shall have $v\dot{y}\times\overline{1+y}^{v-1}=$ $m\dot{y}Ay^{m-1}+n\dot{y}By^{n-1}+p\dot{y}Cy^{p-1}+q\dot{y}Dy^{q-1}$ &c. Whence, multiplying the Sides of the two Equations, cross-wise, and dividing by $\dot{y}\times\overline{1+y}^{v-1}$, there comes out $\overline{1+y}\times mAy^{m-1}+nBy^{n-1}+pCy^{p-1}+qDy^{q-1}$ &c. $=v+vAy^m+vBy^n+vCy^p+vDy^q$ &c. which, by Reduction, is

$$\left.\begin{array}{llllll} mAy^{m-1} & +nBy^{n-1} & +pCy^{p-1} & +qDy^{q-1} & \&c. \\ * & +mAy^m & +nBy^n & +pCy^p & \&c. \\ -v & -vAy^m & -vBy^n & -vCy^p & \&c. \end{array}\right\}=0$$

Now,

Now, since we are at Liberty to take the Exponents of y what we will, so as to answer the Conditions of the Equation, or so that all the Terms here put down may mutually destroy each other; let them, therefore be so taken that the Terms themselves may be homologous, that is, let $m-1=0$, $n-1=m$, $p-1=n$, $q-1=p$ &c. Then, m being $=1$, $n=2$, $p=3$, $q=4$, &c. if these several Values be substituted above, the Equation itself will become

$$\left.\begin{array}{l} A+2By+3Cy^2+4Dy^3+ \text{ \&c.} \\ * +Ay \;\; +2By^2+3Cy^3 \;\; \text{\&c.} \\ -v-vAy-vBy^2-vCy^3 \;\; \text{\&c.} \end{array}\right\}=0$$

Where, taking $A-v=0$, $2B+A-vA=0$, $3C+2B-vB=0$, $4D+3C-vC=0$, &c. so that every Column of homologous Terms (and, consequently, the whole Expression) may vanish, we also get $A=v$; B $\left(=\frac{vA-A}{2}=\frac{A\times\overline{v-1}}{2}\right)=\frac{v\times\overline{v-1}}{2}$; C $\left(=\frac{vB-2B}{3}\right.$ $\left.\frac{B\times\overline{v-2}}{3}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}$; D $\left(=\frac{vC-3C}{4}=C\times\right.$ $\left.\frac{v-3}{4}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, &c. &c.

Whence, by writing these Values, with those of m, n, p, q, &c. in the Series $1+Ay^m+By^n+Cy^p$ &c. first assumed, we, at length, find $\overline{1+y}^v=1+vy+\frac{v}{1}\times\frac{v-1}{2}\times y^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times y^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times y^4+$ &c. which was to be investigated.

From the Series here brought out, any Power or Root, of any other compound Quantity, whether Binomial, Trinomial, &c is easily deduced: For, if p be put to represent the first Term of any such Quantity, and Q the Quotient of the rest of the Terms divided

vided by the first; then the Quantity itself will be expressed by $P+PQ$, or $P\times\overline{1+Q}$, and the v Power thereof by $P^v\times\overline{1+Q}^v$, which therefore is equal to $P^v\times\overline{1+vQ+\frac{v}{1}\times\frac{v-1}{2}\times Q^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times}$ $\overline{Q^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times Q^4+}$ &c., by what is just now determined.

But when v is a Fraction, as in the Notation of Roots, the Theorem here given will be render'd somewhat more commodious for Practice, if, instead of v, a Fraction as $\frac{m}{n}$ be substituted; by which Means it will become $P^{\frac{m}{n}}\times\overline{1+Q}^{\frac{m}{n}}=P^{\frac{m}{n}}\times\overline{1+\frac{m}{n}Q+\frac{m}{n}\times}$ $\overline{\frac{m-n}{2n}Q^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^3+\frac{m}{n}\times\frac{m-n}{2n}\times}$ $\overline{\frac{m-2n}{3n}}\times\frac{m-3n}{4n}Q^4+$ &c. whose Use, in converting radical Quantities into Infinite Serieses will appear from the following Examples.

EXAMPLE VII.

100. *Wherein it is proposed to extract the Square Root of a^2+x^2, in an Infinite Series.*

Here the Quantity to be expanded being $\overline{a^2+x^2}^{\frac{1}{2}}$, or $\overline{aa}^{\frac{1}{2}}\times\overline{1+\frac{xx}{aa}}^{\frac{1}{2}}$, by comparing it with the general Form, $P^{\frac{m}{n}}\times\overline{1+Q}^{\frac{m}{n}}$, we have $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=1$,

and

and $n=2$: Whence, by substituting these Values in the last general Equation, we get

$$\overline{a^2+x^2}|^{\frac{1}{2}} = a \times \overline{1 + \tfrac{1}{2} \times \frac{x^2}{a^2} + \tfrac{1}{2} \times -\tfrac{1}{4} \times \frac{x^4}{a^4} + \tfrac{1}{2} \times -\tfrac{1}{4}}$$

$$\overline{\times -\tfrac{3}{6} \times \frac{x^6}{a^6} + \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times -\tfrac{5}{8} \times \frac{x^8}{a^8}} + \text{\&c.} = a + \frac{x^2}{2a}$$

$- \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Which Series agrees exactly with those found in *Art.* 97. and 98. by different Methods.

EXAMPLE VIII.

101. *Let it be required to extract the Cube-Root of b^3-y^3, in an Infinite Series.*

Here by comparing $\overline{b^3}|^{\frac{1}{3}} \times \overline{1 - \frac{y^3}{b^3}}|^{\frac{1}{3}} \left(= \overline{b^3-y^3}|^{\frac{1}{3}}\right)$ with $P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$, it will be $P=b^3$, $Q=-\frac{y^3}{b^3}$, $m=1$, and $n=3$: Therefore, by Substitution, we get

$$\overline{b^3-y^3}|^{\frac{1}{3}} \left(= b \times \overline{1 - \frac{y^3}{b^3}}|^{\frac{1}{3}}\right) = b \times \overline{1 + \tfrac{1}{3} \times -\frac{y^3}{b^3} + \tfrac{1}{3} \times}$$

$$\overline{-\tfrac{2}{6} \times \frac{y^6}{b^6} + \tfrac{1}{3} \times \tfrac{2}{6} \times -\tfrac{5}{9} \times -\frac{y^9}{b^9} + \tfrac{1}{3} \times -\tfrac{2}{6} \times -\tfrac{5}{9} \times -}$$

$$\overline{\tfrac{8}{12} \times \frac{y^{12}}{b^{12}}} + \text{\&c.} = b - \frac{y^3}{3b^2} - \frac{y^6}{9b^5} - \frac{5y^9}{81b^8} - \frac{10y^{12}}{243b^{11}}$$

&c.

EXAMPLE IX.

102. *Let the Quantity to be converted into an Infinite Series be* $\frac{a}{\sqrt{ax-xx}}$.

In this Case the given Quantity being first transformed to $\sqrt{\frac{a}{x}} \times \overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ and $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ afterwards compared with $\overline{1+Q}|^{\frac{m}{n}}$, we have $Q = -\frac{x}{a}$, $m = -1$, and $n = 2$; and therefore $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ $(=\overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-2n}{2n}Q^2 + \&c.)$ $1 + -\frac{1}{2} \times \frac{-x}{a} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^2}{a^2} + -\frac{1}{2} \times -\frac{3}{4} \times -\frac{5}{6} \times \frac{-x^3}{a^3}$ &c. $= 1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} +$ &c. Which therefore, multiplied by $\sqrt{\frac{a}{x}}$, gives $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}}{8a^{\frac{3}{2}}} + \frac{5x^{\frac{5}{2}}}{16a^{\frac{5}{2}}} + \frac{35x^{\frac{7}{2}}}{128a^{\frac{7}{2}}} +$ &c. $= \frac{a}{\sqrt{ax-xx}}$, the Quantity proposed.

103. It may not be improper to observe here, that, when both the Terms of the proposed Quantity are affirmative, and its Exponent also affirmative and less than Unity, the two first Terms of the equal Series will be positive, and the rest negative and positive, alternately; but if only the first Term of the Binomial be affirmative, all the Terms of the Series, after the first, will be negative: Moreover, if the Exponent of the

the given Quantity be negative, and both the Terms affirmative, the Signs will change alternately; but if only the first be affirmative, all the Terms of the equal Series will be positive.

EXAMPLE X.

104. *Let the Quantity propofed be the Trinomial*

$$\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}}.$$

Here, by dividing the reft of the Terms by the firft, &c. our given Quantity is reduced to $\overline{x^3}|^{\frac{1}{3}} \times \overline{1+2x+3x^2}|^{\frac{1}{3}}$. Therefore, in this Cafe $P=x^3$, $Q=2x+3x^2$, $m=1$, and $n=3$: Whence (by Subftitution) $\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}} = x \times \overline{1+\frac{1}{3}\times\overline{2x+3x^2}+\frac{1}{3}\times -\frac{2}{6}\times\overline{2x+3x^2}|^2+\frac{1}{3}\times-\frac{2}{6}\times-\frac{5}{9}\times\overline{2x+3x^2}|^3}$ &c. $=$

$$x\times\overline{1+\frac{2x+3x^2}{3}-\frac{\overline{2x+3x^2}|^2}{9}+\frac{5\times\overline{2x+3x^2}|^3}{81}}\ \text{\&c.}$$

Which, reduced to fimple Terms, is $= x+\frac{2x^2}{3}+\frac{5x^3}{9}-\frac{68x^4}{81}$ &c.

105. When the propofed Expreffion confifts of a rational, multiply'd by an irrational, Quantity, the Series anfwering to the irrational one muft be firft found, and afterwards multiply'd by the rational Quantity: But, if two, or more, compound irrational Quantities are to be drawn into each other, then take the Series anfwering to each Quantity, feparately, and multiply them together; obferving, always, to neglect all fuch Terms whofe Indices would exceed that of the laft, or higheft,

Term, which the Series sought is proposed to be continued to.

EXAMPLE XI.

106. *Let the Quantity proposed be* $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}}$

First we have $\overline{1-x}|^{\frac{1}{10}} = 1 - \frac{x}{10} - \frac{9x^2}{10 \times 20} - \frac{9 \times 19x^3}{10 \times 20 \times 30} - \frac{9 \times 19 \times 29x^4}{10 \times 20 \times 30 \times 40} -$ &c. Which, multiply'd by $1+x$, produces $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}} = 1 + \frac{9x}{10} - \frac{29x^2}{10.20} - \frac{9.49x^3}{10.20.30} - \frac{9.19.69x^4}{10.20.30.40}$ &c. $= 1 + \frac{9x}{10} - \frac{29x^2}{200} - \frac{147x^3}{2000} - \frac{3933x^4}{80000} -$ &c.

EXAMPLE XII.

107. *Where the Quantity to be expressed in an Infinite Series is* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, or $\overline{a^2-x^2}|^{\frac{1}{2}} \times \overline{c^2-x^2}|^{-\frac{1}{2}}$.

Here we have, $\overline{a^2-x^2}|^{\frac{1}{2}}$ $(a \times \overline{1-\frac{xx}{aa}}|^{\frac{1}{2}} = a \times$
$\overline{1 + \frac{1}{2} \times -\frac{x^2}{a^2} + \frac{1}{2} \times -\frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times -\frac{x^6}{a^6} + \&c.} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

And

And $\overline{c^2 - x^2}|^{-\frac{1}{2}}$ $(= c^{-1} \times \overline{1 - \frac{xx}{cc}}|^{-\frac{1}{2}} = c^{-1} \times$ $\overline{1 + -\frac{1}{2} \times -\frac{x^2}{c^2} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^4}{c^4} + \text{\&c.}} = \frac{1}{c} +$ $\frac{x^2}{2c^3} + \frac{3x^4}{8c^5} + \frac{5x^6}{16c^7}$ &c. Whence, multiplying these two Values, one by the other, we get

$$\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 + \text{\&c.}$$

for the four first Terms of the Series sought.

EXAMPLE XIII.

108. *Let the Quantity to be expanded be the Multinomial, or Infinite Series,* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \text{\&c.}}|^v$; *whose Exponent v denotes any Number whatever, whole or broken, positive or negative.*

Here, dividing by the first Term, the given Quantity is transformed to $x^{pv} \times \overline{1 + ax^n + bx^{2n} + cx^{3n} + dx^{4n} + \text{\&c.}}|^v$; which, if $ax^n + bx^{2n} + cx^{3n}$ &c. be put $= y$, will become $x^{pv} \times \overline{1+y}|^v$; which last Expression (by *Art.* 99.) is $=$ $x^{pv} \times \overline{1 + vy + \frac{v}{1} \times \frac{v-1}{2} \times y^2 + \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3} \times y^3 + \text{\&c.}}$ Whence (for Brevity sake) putting $A = v$, $B = \frac{v}{1} \times \frac{v-1}{2}$, $C = \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = \frac{v}{1} \times$

$\frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, *&c.* and substituting for y, there comes out $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}^{\,v} =$ $x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} \times x^{pv+2n} +$ $\overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4}$ $\times x^{pv+4n} + \overline{Ae + 2Bad + 2Bbc + 3Ca^2c + 3Cab^2 + 4Da^3b}$ $\overline{+ Ea^5} \times x^{pv+5n} +$ *&c.*

EXAMPLE XIV.

109. *To extract the Square Root of $a^2 - x^2$, and from thence to determine the Fluent of $\dot{x}\sqrt{a^2 - x^2}$, in an Infinite Series.*

By proceeding as in the foregoing Examples, the Value of $\sqrt{a^2 - x^2}$ in an Infinite Series will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} -$ *&c.* Which multiply'd by $\dot{x}$ gives $\dot{x}\sqrt{a^2 - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ *&c.* Whose Fluent therefore (by *Art.* 77.) is $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} \frac{5x^9}{1152a^7} -$ *&c.* Which was to be determined.

EXAMPLE XV.

110. *Let it be required to approximate the Fluent of*

$$\frac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x^n \dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$$ *in an Infinite Series.*

It appears, from Example 12, that the Value of $\frac{\overline{a^2 - x^2}|^{\frac{1}{2}}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$, expressed in a Series, is $\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 +$ &c. Which Value being therefore multiply'd by $x^n \dot{x}$, and the Fluent taken (by the common Method) we get $\frac{ax^{n+1}}{n+1 \times c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times \frac{x^{n+3}}{n+3} + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times \frac{x^{n+5}}{n+5} + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times \frac{x^{n+7}}{n+7} +$ &c.

EXAMPLE XVI.

111. *Wherein it is proposed to approximate the Fluent of* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v \times x^{m-1}\dot{x}$ *in a Series.*

Here, if A be put $=v$, $B=v\times\frac{v-1}{2}$, $C=v\times\frac{v-1}{2}\times\frac{v-2}{3}$, $D=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, &c. the Quantity $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n}\ \&c.}|^v$ expanded, will be $= x^{pv} + Aax^{pv+n} + \overline{Ab+Ba^2} + x^{pv+2n} + \overline{Ac+2Bab+Ca^3}\times x^{pv+3n} + \overline{Ad+2Bac+Bb^2+3Ca^2b+Da^4}\times x^{pv+4n} + \&c.$ as appears from *Art.* 108. Therefore this Expression being multiply'd by $x^{m-1}\dot{x}$, and the Fluent taken (as usual) we shall have $\frac{x^{pv+m}}{pv+m} +$

$$\frac{Aax^{pv+m+n}}{pv+m+n} + \frac{\overline{Ab+Ba^2}\times x^{pv+m+2n}}{pv+m+2n} + \frac{\overline{Ac+2Bab+Ca^3}\times x^{pv+m+3n}}{pv+m+3n} + \frac{\overline{Ad+2Bac+Bb^2+3Ca^2b+Da^4}\times x^{pv+m+4n}}{pv+m+4n} + \&c.$$

for the Quantity proposed to be found.

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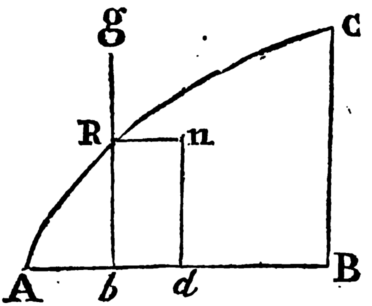
SECTION VII.

Of the Use of Fluxions in finding the Areas of Curves.

CASE I.

112. LET ARC *be a Curve of any Kind whose Ordinates are perpendicular to an Axis* AB.

Imagine a Right-line *b*R*g* (perpendicular to AB) to move parallel to itself from A towards B; and let the Celerity thereof, or the Fluxion of the Abscissa A*b*, in any proposed Position of that Line, be denoted by *bd*: Then it will appear, *from Art.* 4. that the Rectangle (*bn*) under *bd* and the Ordinate *b*R, will express the corresponding Fluxion of the generated Area *ab*R: Which Fluxion, if $Ab=x$, and $bR=y$, will therefore be



$=y\dot{x}$: From whence, by substituting for y or $\dot{x}$ (according to the Equation of the Curve) and taking the Fluent, the Area itself will become known.

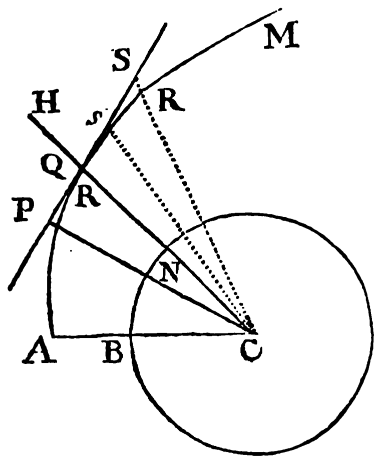
CASE II.

113. *Let* ARM *be any Curve whose Ordinates* CR, CR *are all referred to a Point or Center.*

Conceive a Right-line CRH to revolve about the given Center C, and let a Point R move along the said

said Line, so as to trace out, or describe the proposed Curve Line ARM.

Now it is evident, that, if the Point R was to move from any Position Q, without changing its Direction and Velocity, it would proceed along the Tangent QS (instead of the Curve) and describe Areas QsC, QSC about the Center C, proportional to the Times of their Description; because those Areas, or Triangles, having the same Altitude (CP), are as the Bases Qs and QS, and these are as the Times, because the Motion in the Tangent (upon that Supposition) would be uniform.



Hence, if RS be taken to denote the Value of ($\dot{z}$) the Fluxion of the Curve Line AR, the corresponding Fluxion of the Area ARC, will be truly represented by the, uniformly generated, Triangle QCS *: Which, putting the Perpendicular (CP) drawn from the Center to the Tangent, $=s$, will therefore be ($=\frac{QS\times CP}{2}=\frac{s\dot{z}}{2}$); from whence the Area itself may be determined.

* Art. 2 and 5.

But, since in many Cases, the Value of $\dot{z}$ cannot be computed (from the Property of the Curve) without some Trouble, the two following Expressions, for the Fluxion of the Area, will commonly be found more commodious, *viz:* $\frac{sy\dot{y}}{2t}$ and $\frac{y^2\dot{x}}{2a}$; where $t=$ RP and $x=$ the Arch BN of a Circle, described about the Center C, at

any

any Distance a ($=$CB). These Expressions are derived from that above, in the following Manner; *viz.*

$\dot{z}:\dot{y}::y$ (CR) : t (RP) *; therefore $\dot{z}=\frac{y\dot{y}}{t}$; and * Art. 35.

consequently $\frac{s\dot{z}}{2}=\frac{sy\dot{y}}{2t}$; which is the first Expression.

Again, because the Celerity of R in the Direction of the Tangent is denoted by $\dot{z}$, that in a Direction perpendicular to CQ (whereby the Point R revolves about the Center C) will therefore be ($=\frac{\text{CP}}{\text{CR}}\times\dot{z}$) * $=$ * Art. 35.

$\frac{s\dot{z}}{y}$; which being to ($\dot{x}$) the Celerity of the Point N (about the same Center) as the Distance (or Radius) CR (y) to the Radius CN (a) we shall, by multiplying Extremes and Means, have $\frac{as\dot{z}}{y}=y\dot{x}$; and consequently

$\frac{s\dot{z}}{2}=\frac{y^2\dot{x}}{2a}$; which is the other Expression.

The Method of applying this, together with the preceding Forms, will appear at large from the following Examples: Wherein x, y, z, and u are all along put to denote the Abscissa, Ordinate, Curve-line, and the Area respectively, unless where the contrary is expressly specified.

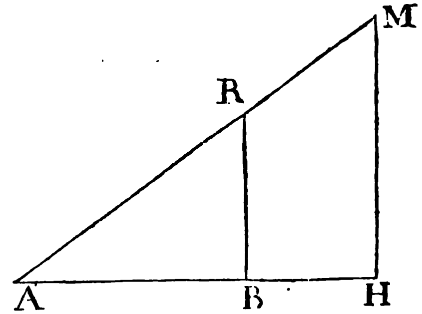
EXAMPLE I.

114. *Let it be proposed to determine the Area of a right-angled Triangle* AHM.

Put the Base AH$=a$, the Perpendicular HM$=b$; and let AB (x) be any Portion of the Base, considered as a flowing Quantity, and let BR (y) be the Ordinate, or Perpendicular, corresponding:

Then,

Then, becauſe of the ſimilar Triangles AHM and ABR, it will be, $a : b :: x : y = \frac{bx}{a}$: Whence $y\dot{x}$



* Art. 112. (the Fluxion of the Area ABR *) is, in this Caſe, $= \frac{bx\dot{x}}{a}$; and conſequently the Fluent thereof, or the Area

† Art. 77. itſelf $= \frac{bx^2}{2a}$ †: Which therefore, when $x = a$, and BR coincides with HM, will become $\frac{ab}{2} = \frac{AH \times HM}{2} =$ the Area of the whole Triangle AHM; which we alſo know from other Principles.

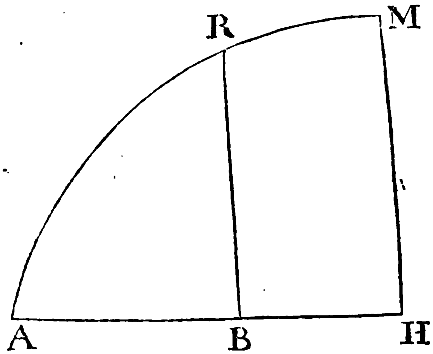
EXAMPLE II.

115. *Let the Curve* ARMH, *whoſe Area you would find, be the common Parâbola.*

In which Caſe the Relation of AB (x) and BR (y) being expreſſed by $y^2 = ax$ (where a is the Parameter)

‡ Art. 112. we thence get $y = a^{\frac{1}{2}} x^{\frac{1}{2}}$; and therefore $\dot{u}$ $(= y\dot{x}$ ‡$)$ $= a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x}$: Whence $u = \frac{2}{3} \times a^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} a^{\frac{1}{2}} x^{\frac{1}{2}} \times x = \frac{2}{3} yx$ (becauſe

(because $a^{\frac{1}{2}}x^{\frac{1}{2}}=y$) $= \frac{2}{3} \times AB \times BR$: Hence *a Parabola is* $\frac{2}{3}$ *of a Rectangle of the same Base and Altitude.*



The Area is here found in Terms of x; but it will, many times, be more easily brought out in Terms of y (without radical Quantities) as in the very Case last proposed: Where, because $x = \frac{y^2}{a}$, we therefore have $\dot{x} = \frac{2y\dot{y}}{a}$; and consequently $\dot{u}\ (y\dot{x}) = \frac{2y^2\dot{y}}{a}$: Whence $u = \frac{2y^3}{3a} = \frac{2y}{3} \times \frac{y^2}{a} = \frac{2y}{3} \times x = \frac{2}{3} \times AB \times BR$; *the same as before.*

EXAMPLE III.

116. *Let* ARM (*see the preceding Figure*) *be a Parabola of any Kind; whereof the general Equation is* $y^{m+n} = a^m x^n$.

Therefore, by extracting the Root, or dividing each Exponent by $m+n$, we have $y = a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}}$; whence $\dot{u}$

$\dot{u}\ (y\dot{x}) = a^{\frac{m}{m+n}} \times \dot{x}x^{\frac{n}{m+n}}$; and consequently u (the true Fluent, or Area) $= a^{\frac{m}{m+n}} \times \dfrac{x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} =$

$$\frac{a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}} \times x \times \overline{m+n}}{m+2n} = \frac{m+n}{m+2n} \times yx = \frac{m+n}{m+2n} \times$$

AB×BR.

No Notice has been yet taken of any constant Quantity to be added to, or subtracted from, the variable One, first found, in order to render it complete, agreeable to the Observation in *Art.* 78.

But that no such Correction is required in any of the preceding Examples, is evident from the Nature of the Figure; because, when x and y are nothing, the Area (u) ought also to be nothing, which it actually is according to the Equations above exhibited.

The Fluent found in the succeeding Example, will, however, stand in need of a Correction.

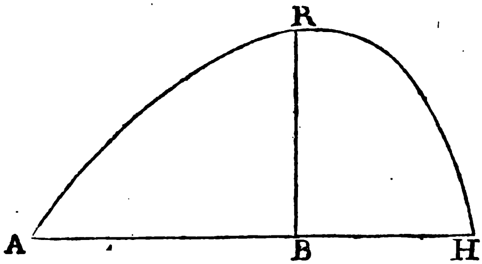
EXAMPLE IV.

117. *Where it is proposed to find the Area of the Curve* ARH, *whose Equation is* $x^4 - a^2x^2 + a^2y^2 = 0$.

Here, the given Equation is reduced to $y = \dfrac{x \times \overline{a^2 - x^2}^{\frac{1}{2}}}{a}$; whence $\dot{u}\ (= y\dot{x}) = \dfrac{\overline{a^2 - x^2}^{\frac{1}{2}} \times x\dot{x}}{a}$:

Whereof the Fluent (by the common Rule *) is —

* Art. 77.



$\dfrac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$: Which, when $x=0$ and $u=0$, becomes $-\dfrac{a^2}{3}$; this therefore subtracted from $-\dfrac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$, leaves $\dfrac{a^2}{3}-\dfrac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$ for the Fluent corrected, or the true Value of the Area ABR *.

* Art. 78.

When the Ordinate BR $\left(\dfrac{x\sqrt{a^2-x^2}}{a}\right)$ becomes equal to Nothing, and B coincides with H, then x will become $=a=$ AH; and therefore the Area of the whole Curve ARH will be barely $=\dfrac{a^2}{3}=\frac{1}{3}\text{AH}^2$.

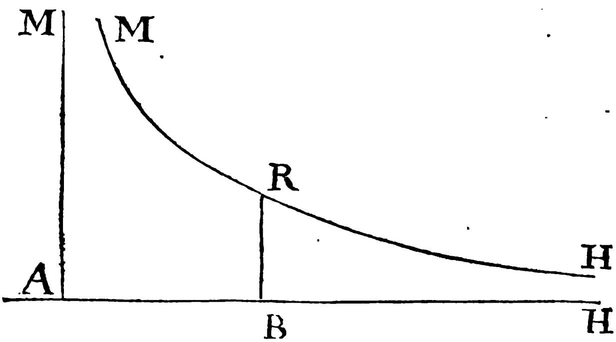
EXAMPLE V.

118. *Let it be required to determine the Area of the hyperbolical Curve whose Equation is* $x^m y^n = a^{m+n}$.

In this Case we have $y=\dfrac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}}=a^{\frac{m+n}{n}}\times x^{\frac{-m}{n}}$;

and

and therefore $\dot{u}$ $(=y\dot{x}) = a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}}\dot{x}$: Whofe Fluent is $\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}} = \frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}$; which, when x is



$=0$, will alfo be $=0$, if n be greater than m: Therefore, the Fluent requires no Correction in this Cafe; the Area AMRB. included between the Afymptote AM and the Ordinate BR, being truly defined by $\left(\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}\right)$ the Quantity above determined. But, if n be lefs than m, then the Fluent, when $x=0$, will be infinite (becaufe the Index $\frac{n-m}{n}$ being negative, 0 becomes a Divifor to $na^{\frac{m+n}{n}}$:) Whence the Area AMRB will alfo be infinite.

But, here, the Area BRH comprehended between the Ordinate, the Curve, and the Part BH of the other Afymptote, is finite, and will be truly expounded by $na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}$, the fame Quantity with its Signs changed. For the Fluxion

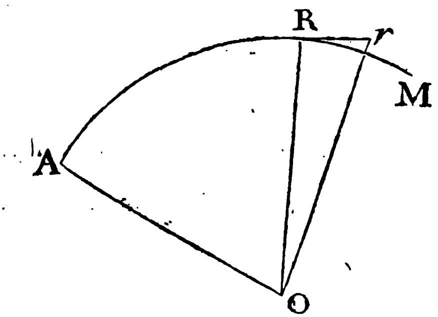
Fluxion of the Part AMRB being $a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$, that of its Supplement BRH must consequently be $-a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$: Whereof the Fluent is $-\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}}$

$= \frac{a^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ = the Area BRH: Which wants no Correction; because, when x is infinite, and the Area BRH = 0, the said Fluent will also intirely vanish, seeing the Value of $x^{\frac{m-n}{n}}$ (which is a Divisor to $a^{\frac{m+n}{n}}$) is then infinite.

EXAMPLE VI.

119. *Where let it be required to determine the Area of the circular Sector* AOR.

Then, putting the Radius AO (or OR) = a, the



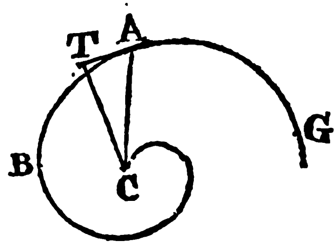
Arch AR (considered as variable by the Motion of R) = x, and Rr = $\dot{x}$, the Fluxion of the Area will here

* Art. 113. be expressed by $\frac{a\dot{z}}{2}$ (=the Triangle ORr * :) Whence the Area itself is $= \frac{az}{2} = AO \times \frac{1}{2}AR$: From which it appears that the Area of any Circle is expressed by a Rectangle under half the Circumference and half the Diameter.

EXAMPLE VII.

120. *Wherein it is proposed to determine the Area* CBAC *of the logarithmic Spiral.*

Let the Right-line AT touch the Curve at A; upon which, from the Center C, let fall the Perpendicular CT: Then, since by the Nature of the Curve the



Angle TAC is every where the same, the Ratio of AT (t) to CT (s) will here be constant: And therefore the

* Art. 113. Fluent of $\frac{s}{t} \times \frac{y\dot{y}}{2}$ * $= \frac{s}{t} \times \frac{y^2}{4} =$ the Area which was to be found.

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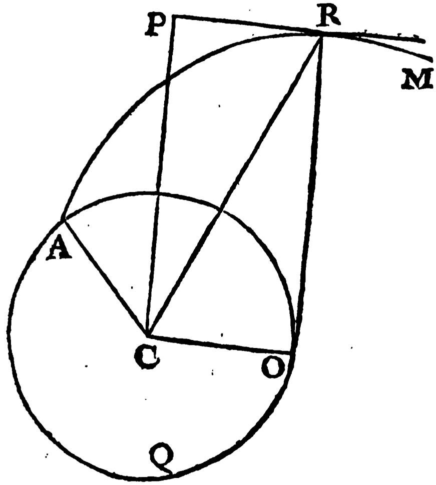
EXAMPLE VIII.

121. *Let the Curve* ARM *be the Involute of a given Circle* AOQ.

In which Caſe the intercepted Part of the Tangent RP (t) being every where equal to the Radius CO (a)



of the generating Circle, we therefore have CP (s) $=$ $\sqrt{CR^2 - RP^2}$) $= \sqrt{y^2 - a^2}$: Whence $\dot{u}$ $(= \frac{sy\dot{y}}{2t}$ *) * Art. 113.

$= \frac{\sqrt{y^2 - a^2} \times y\dot{y}}{2a}$; and conſequently $u = \frac{\overline{y^2 - a^2}^{\frac{3}{2}}}{6a} =$ $\frac{CP^3}{6CA} =$ the required Area ACR:

Which will alſo expreſs the Area ARO generated by the Radius of Evolution RO; becauſe, RO being $=$

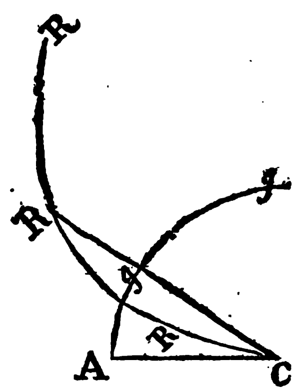
 the

* Art. 119. the Arch AO, the Sector ACO ($\frac{1}{2}$ AO × OC *) is equal to the Triangle CRO ($\frac{1}{2}$ RO×OC) which equal Quantities being successively subtracted from CARO, there remains AOR=ACR.

EXAMPLE IX.

122. *Let the Curve* CRR, *whose Area* CRgC *you would find, be the Spiral of* Archimedes.

Let AC be a Tangent to the Curve at the Center



C, about which Center, with any Radius AC ($=a$) suppose a Circle Agg to be described; then the Arch (or Abscissa) Ag corresponding to any proposed Ordinate CR, being to that Ordinate in a given, or constant, Ratio (suppose as m to n) we have x (Ag) $=$ * Art. 113. $\frac{my}{n}$; therefore $\dot{u} = \frac{y^2\dot{x}}{2a}$ * $= \frac{my^2\dot{y}}{2an}$, and consequently u $= \frac{my^3}{6an} =$ the Area CRRgC.

EXAMPLE X.

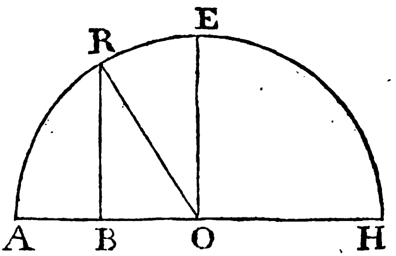
123. *Let the Equation of the Spiral* CRR (*see the last Figure*) *be* $x = by + cy^2 + dy^3 + ey^4 + fy^5 +$ &c.

Then, $\dot{x}$ being $= b\dot{y} + 2cy\dot{y} + 3dy^2\dot{y} + 4ey^3\dot{y} +$ &c. we shall have $\dot{u}\ \left(= \frac{y^2\dot{x}}{2a}\right) = \frac{by^2\dot{y}}{2a} + \frac{2cy^3\dot{y}}{2a} + \frac{3dy^4\dot{y}}{2a}$ $+ \frac{4ey^5\dot{y}}{2a} +$ &c. and therefore $u = \frac{by^3}{6a} + \frac{2cy^4}{8a} + \frac{3dy^5}{10a} + \frac{4ey^6}{12a}$ &c. = the true Value of the Area in this Case.

EXAMPLE XI.

124. *Let it be proposed to find the Area of a Semicircle* AREH.

Here, putting the Diameter $AH = a$, $AB = x$, and $BR = y$ &c. (*as usual*) we have y^2 (BR^2) $= ax - xx$



($AB \times BH$), and consequently $\dot{u}$ ($y\dot{x}$) $= \dot{x}\sqrt{ax - xx} =$ $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{a}}\Big|^{\frac{1}{2}}$: Which Expression not being of the Kind described in *Art.* 83 and 85. that admit of Fluents in finite

finite Terms, let it therefore be resolved into an Infinite Series *, and you will have $\dot{z} = a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} \times \overline{1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4}}$ &c. $= a^{\frac{1}{2}} \times \overline{x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^2} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^3}}$ &c. From whence, the Fluent of every Term being taken, according to the common Method, there will come out $z = a^{\frac{1}{2}} \times \overline{\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5a} - \frac{x^{\frac{7}{2}}}{28a^2} - \frac{x^{\frac{9}{2}}}{72a^3} - \frac{5x^{\frac{11}{2}}}{704a^4}}$ &c. $= x\sqrt{ax} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{x^3}{72a^3} - \frac{5x^4}{704a^4}}$ — &c. = the Area ABR. Now, when $x = \frac{1}{2}a$, the Ordinate BR will coincide with the Radius OE; in which Case the Area becomes $= \frac{1}{2}a\sqrt{\frac{1}{2}aa} \times \overline{\frac{2}{3} - \frac{1}{10} - \frac{1}{112} - \frac{1}{576} - \frac{5}{11264}}$ &c. $= \frac{a^2\sqrt{\frac{1}{2}}}{2} \times \overline{0,6666 - 0,1 - 0,0089 - 0,0017 - 0,0004}$ &c. $= 0,1964a^2$; which, multiply'd by 2, gives $0,3928a^2$ for the Area of the Semi-circle AEH, nearly.

* Art. 90 and 99.

As the foregoing Series, in finding the Area of the whole Quadrant AOE, converges but slowly, a considerable Number of Terms ought therefore to be taken to have the Conclusion but tolerably exact, the five first Terms above collected being sufficient to bring out no more than three Places of Figures that can be depended on. For which Reason it may be of Use to consider, whether, by computing the Area of some particular Portion (ABR) of the said Quadrant, that of the whole may not be deduced; where x being small in com-

comparifon of a, the Series may have fuch a Rate of Convergency, that a fmaller Number of Terms will be fufficient *.

* Art. 92.

Now, in order to this, it is well known that, if the Arch AR be taken $= \frac{1}{3}$ AE (or 30 Degrees) the Sine BR will be $= \frac{1}{2}$ AO; and confequently AB (x) = AO — OB = AO — $\sqrt{OR^2 - BR^2}$; which, if the Radius AO be expounded by Unity, (to facilitate the Operation) will be = 0,1339746 very nearly: This therefore, with the Value of a, being fubftituted in the forementioned Series, $\sqrt{ax^3} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \&c.}$ we have $\overline{0{,}0693505} \times \overline{0{,}6666666 - 0{,}0133975 - 0{,}0001603 - 0{,}0000042 - \&c.} = 0{,}0693505 \times 0{,}6531046 = 0{,}0452931$ = the Area ABR: Which added to the Area OBR $(= OB \times \frac{1}{2} BR = \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0{,}2165063)$ gives 0,2617994, for the Area of the Sector AOR; the treble whereof, or 0,7853982 (becaufe AR $= \frac{1}{3}$AE) will therefore be the Content of the whole Quadrant AOE: Which Number, found by taking four Terms of the Series only, is true to the laft Decimal Place.

This Conclufion may be otherwife brought out, by finding a Series for the other Part of the Area, included between the Radius OE and the Ordinate BR; wherein the Co-fine OB (inftead of the verfed Sine AB) will be the converging (or variable) Quantity.

For, putting OB $= x$, and OR (OA) $= b$, we have y $(BR = \sqrt{OR^2 - OB^2} = \overline{b^2 - x^2}|^{\frac{1}{2}}$; and confequently $(y\dot{x})$ the Fluxion of the Area OBRE * $= \dot{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}} = b\dot{x} - \frac{x^2\dot{x}}{2b} - \frac{x^4\dot{x}}{8b^3} - \frac{x^6\dot{x}}{16b^5} - \frac{5x^8\dot{x}}{128b^7} - \frac{7x^{10}\dot{x}}{256b^9}$ &c. Whence the Area itfelf is $= bx - \frac{x^3}{6b} - \frac{x^5}{40b^3} - \frac{x^7}{112b^5} - \frac{5x^9}{1152b^7} - \frac{7x^{11}}{2816b^9}$ &c.

* Art. 112.

Now, if $\dot{x}$ (OB) be assumed $= \frac{1}{2}$ AO (so that the Arch ER may be $= \frac{1}{3}$ AE) and the Value of b (AO) be expounded by Unity, we shall have

$$x^3 \left(= x \times x^2 = ,5 \times \frac{1}{4} = \frac{,5}{4}\right) = ,125$$

$$x^5 \left(= x^3 \times x^2 = \frac{,125}{4}\right) = ,03125$$

$$x^7 \left(= x^5 \times x^2 = \frac{,03125}{4}\right) = ,0078125$$

$$x^9 \left(= x^7 \times x^2 = \frac{x^7}{4}\right) = ,0019531+$$

$$x^{11} \left(= x^9 \times x^2 = \frac{x^9}{4}\right) = ,0004883-$$

&c.

Which Values of the Powers of x being respectively divided by 6, 40, 112, 1152, 2816, &c. there will result 0,5000000 — 0,0208333 — 0,0007812 — 0,0000698 — 0,0000085 — 0,0000012 — 0,0000002 &c. = 0,4783057, for the Area OBRE in the forementioned Circumstance, when OB $= \frac{1}{2}$ OA: From which, deducting the Triangle OBR ($= \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0,2165063$) the Remainder ,2617994 will consequently be the Area of the Sector EOR; the treble whereof (because ER is, *here*, $= \frac{1}{3}$ AE) will give the Area of the whole Quadrant, 0,7853982; *as before.*

EXAMPLE XII.

125. *Let the Curve, whose Area you would find, be the Cissoid of Diocles; whereof the Equation is* $y^2 = \frac{x^3}{a-x}$.

* Art. 112. Here we have u ($y\dot{x}$ *) $= \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}} \times \overline{1-\frac{x}{a}}\big|^{\frac{1}{2}}}$

$= \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$: Which being none of the Kind that admit of Fluents in finite Terms*, let it therefore be resolved into an Infinite Series, and you will have $\dot{u} =$

* Art. 83 and 85.

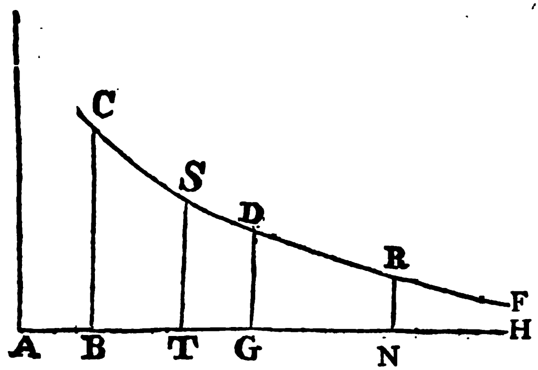
$$\frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} + \&c.} = \frac{1}{a^{\frac{1}{2}}} \times \overline{x^{\frac{3}{2}}\dot{x} + \frac{x^{\frac{5}{2}}\dot{x}}{2a} + \frac{3x^{\frac{7}{2}}\dot{x}}{8a^2} + \frac{5x^{\frac{9}{2}}\dot{x}}{16a^3} + \&c.}$$

Whence u (the Area itself) will come out $= \frac{1}{a^{\frac{1}{2}}} \times \overline{\frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{7}{2}}}{7a} + \frac{x^{\frac{9}{2}}}{12a^2} + \frac{5x^{\frac{11}{2}}}{88a^3} + \&c.} = x^2\sqrt{\frac{x}{a}} \times \overline{\frac{2}{5} + \frac{x}{7a} + \frac{x^2}{12a^2} + \frac{5x^3}{88a^3} + \&c.}$

EXAMPLE XIII.

126. *Let the proposed Curve* CSDR *be of such a Nature, that (supposing* AB *Unity) the Sum of the Areas* CSTBC *and* CDGBC *answering to any two proposed Abscissas* AT *and* AG, *shall be equal to the Area* CRNBC *whose corresponding Abscissa* AN *is equal to,* AT×AG, *the Product of the Measures of the two former Abscissas.*

First, in order to determine the Equation of the Curve, (which must be known before the Area can be found) let the Ordinates GD and NR move parallel to themselves towards HF; and, then, having put $GD = y$, $NR = z$, NR$=z$, AT$=a$, AG$=s$, and AN$=u$, the Fluxion of the Area CDGB will be represented by $y\dot{s}$, and that



* Art. 112. of the Area CRNB by $z\dot{u}$ *: Which two Expressions must, by the Nature of the Problem, be equal to each other; because the latter Area CRNB exceeds the former CDGB by the Area CSTB, which is here considered as a constant Quantity; and it is evident that two Expressions, that differ only by a constant Quantity, must always have equal Fluxions.

Since, therefore $y\dot{s}$ is $=z\dot{u}$, and $u=as$, *by Hypothesis*, it follows that $\dot{u}=a\dot{s}$, and that the first Equation (by substituting for $\dot{u}$) will become $y\dot{s}=az\dot{s}$, or $y=az$, or lastly $ys=zas$, that is, GD×AG=NR×AN: Therefore GD : NR :: AN : AG; whence it appears that every Ordinate of the Curve is reciprocally as its corresponding Abscissa.

Now, to find the Area of the Curve so determined, put BC $=b$, and BG $=x$: Then, since AG $(1+x)$: AB (1) :: BC (b) : GD (y) we have $y=\frac{b}{1+x}$, and consequently $\dot{u}\ (=y\dot{x})=\frac{b\dot{x}}{1+x}=b\times\overline{\dot{x}-x\dot{x}+x^2\dot{x}-x^3\dot{x}+x^4\dot{x}-}$ &c.. Whence, BGDC, the Area itself

ſelf will be $= b \times x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. Which was to be found.

It may be here obſerved that the Areas of the Spaces above mentioned, are analogous to, and have the very ſame Properties as *Logarithms*; and that thoſe Spaces, or Logarithms, may be of different Forms or Values, according as you take the Value of the firſt Ordinate BC, which may be aſſumed at Pleaſure: Thus, if BC be taken $=$AB$=$Unity, the Curve will become an equilateral Hyperbola whoſe Center is A (becauſe then AG $\times$GD$=$AB^2) and in that Caſe they are called hyperbolical Logarithms: But, if BC be taken$=$0,43429448 (ſo that the Logarithm, or the Area of the Space CDGB, anſwering to the Abſciſſa AG, when expreſſed by the Number 10, may be expounded by Unity, or AB^2) we ſhall then have the common, or *Brigean* Form of Logarithms.

From theſe Logarithms (given by the Tables) the Buſineſs of finding Fluents, is in many Caſes, very much facilitated: For, if the Fluxion given appears to agree with the Fluxion of any known Logarithmic Expreſſion, its Fluent may, it is evident, be had by the Tables, ready calculated, without the Trouble of an Infinite Series.

But, now to know what Kinds of Fluents are explicable by Means of Logarithms, it will be neceſſary to obſerve that, *the Fluxion of any hyperbolic Logarithm is always expreſſed by the Fluxion of the correſponding Number, divided by that Number*; This appears from above, where $(y\dot{x})$ the Fluxion of the Area (or Logarithm) BGDC, when BC$=$AB$=1$, is truly repreſented by $\frac{\dot{x}}{1+x}$; where $1+x$ ($=$AG) may ſtand for any Number whatever; and $\dot{x}$ for its Fluxion.

Hence

Hence the Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expressed by the hyperbolical Logarithm of $x+\sqrt{x^2 \pm a^2}$: For the Fluxion of $(x+\sqrt{x^2 \pm a^2})$ the Number itself, being $\dot{x}$ $+ \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}\sqrt{x^2 \pm a^2}+x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ $\times \overline{\sqrt{x^2 \pm a^2}+x}$, this last Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{\dot{x}^2 \pm a^2}}$, the very Fluxion first proposed.

It also appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax+x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a+x+\sqrt{2ax+x^2}$: Because the Fluxion of the Number $(a+x+\sqrt{2ax+x^2})$ is here $= \dot{x} + \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} =$ $\frac{\dot{x}}{\sqrt{2ax+xx}} \times \overline{\sqrt{2ax+xx}+a+x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax+xx}}$.

Likewise the Fluent of $\frac{2a\dot{x}}{a^2-x^2}$ will be represented by the hyperbolical Logarithm of $\frac{a+x}{a-x}$: Because, the Fluxion of $\frac{a+x}{a-x}$, being $\frac{\dot{x}\times\overline{a-x}+\dot{x}\times\overline{a+x}}{\overline{a-x}|^2} = \frac{2a\dot{x}}{\overline{a-x}|^2}$, if the same be therefore divided by $\frac{a+x}{a-x}$, we shall have $\frac{2a\dot{x}}{\overline{a-x}|^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{a-x\times a+x} = \frac{2a\dot{x}}{a^2-x^2}$.

Lastly,

Lastly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ will be denoted by the hyperbolical Logarithm of $\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2 \pm x^2}} \times$

$$\frac{\overline{a+\sqrt{a^2 \pm x^2}}}{\overline{a+\sqrt{a^2 \pm x^2}}\big|^2} \mp \frac{x\dot{x}}{\sqrt{a^2 \pm x^2}} \times \frac{\overline{a-\sqrt{a^2 \pm x^2}}}{\overline{a+\sqrt{a^2 \pm x^2}}\big|^2} =$$

$$\frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \overline{a+\sqrt{a^2 \pm x^2}}\big|^2}$$; which divided by

$\frac{a-\sqrt{a^2 \pm x^2}}{a+\sqrt{a^2 \pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \overline{a+\sqrt{a^2 \pm x^2}}\big|^2} \times$

$$\frac{a+\sqrt{a^2 \pm x^2}}{a-\sqrt{a^2 \pm x^2}} = \frac{\pm 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times a+\sqrt{a^2 \pm x^2} \times a-\sqrt{a^2 \pm x^2}}$$

$= \frac{\mp 2ax\dot{x}}{\sqrt{a^2 \pm x^2} \times \mp x^2} = \frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$, the Fluxion proposed.

These four are the principal Forms of Fluxions; whose Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occasion, may be easily supply'd by a Table of the common Form: For, since the hyperbolical Logarithm of any Number is to the common Logarithm of the same Number, in the constant Ratio of Unity to 0,43429448 (as appears from above) it follows that if any common Logarithm be, either, divided by 0,43429448, or multiply'd by its Reciprocal 2,30258509, you will thence obtain the hyperbolical Logarithm corresponding.

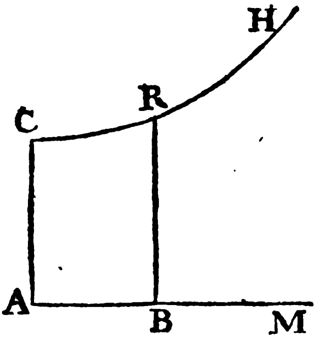
EX-

EXAMPLE XIV.

127. *Let it be required to determine the Area of the Curve; whose Equation is* $a^2y-x^2y-a^3=0$.

In which Case y being $=\frac{a^3}{a^2-x^2}$, we have $\dot{u}$ $(=y\dot{x})$* $=\frac{a^3\dot{x}}{a^2-x^2}=a\dot{x}+\frac{x^2\dot{x}}{a}+\frac{x^4\dot{x}}{a^3}+\frac{x^6\dot{x}}{a^5}+\frac{x^8\dot{x}}{a^7}+$ &c.

* Art. 112.



Whence $u=ax+\frac{x^3}{3a}+\frac{x^5}{5a^3}+\frac{x^7}{7a^5}+\frac{x^9}{9a^7}+$ &c. $=$ the Area sought.

But the same Area (or Fluent) may be found without an Infinite Series, by Means of a Table of Logarithms, agreeable to the Observations in the last Article: For, since it there appears that the Fluent of $\frac{2a\dot{x}}{a^2-x^2}$ is truly expressed by the hyperbolic Logarithm of $\frac{a+x}{a-x}$, it follows that that of $\frac{a^3\dot{x}}{a^2-x^2}$ $\left(=\frac{2a\dot{x}}{a^2-x^2}\times\frac{1}{2}a^2\right)$ will be expressed by the same Logarithm multiply'd by $\frac{1}{2}a^2$. Thus, for Example sake, let a $(=AC)$ be taken

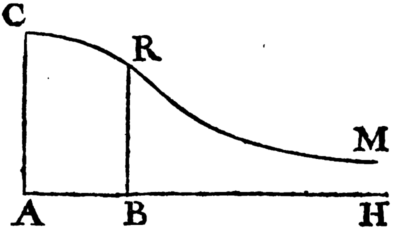
taken = 10, and x (=AB) = 5; then will $\frac{a+x}{a-x}=3$; whoſe Logarithm taken from the common Tables is 0,4771213; which multiply'd by the *Modulus* 2,30258509 (ſee the laſt Article) gives 1,09861228 for the hyperbolical Logarithm of $\frac{a+x}{a-x}$; and this again multiply'd by 50 ($\frac{1}{2}a^2$) produces 54,930614 for the true Value of the Area ABRC, in the aforeſaid Circumſtance, when AC=10, and AB=5.

EXAMPLE XV.

128. *Where the propoſed Curve is that whoſe Equation is* $a^2y^2+x^2y^2=a^4$.

Here, by reducing the given Equation, we get $y=\frac{a^2}{\sqrt{a^2+x^2}}$: Therefore $y\dot{x}=\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}=\dot{u}$ *. * Art. 111.

Whence, the Fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being = hyperb.



Log. of $x+\sqrt{a^2+x^2}$ (by *Art.* 126.) that of $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ will conſequently be = the ſame Logarithm multiply'd by a^2.

But to find whether the Fluent thus determined does not need a Correction ‡, let x be taken = 0; then the ‡ Art. 78. Fluent

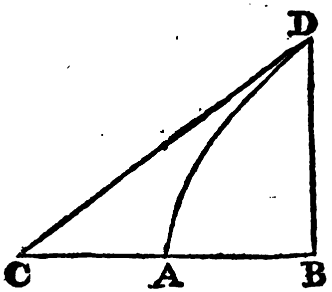
Fluent will become $=$ hyp. Log. $a : \times a^2$: Which, therefore, must be substracted, to have the true Value of the Area ACRB *; and then there results $a^2 \times$ hyp. Log. $x + \sqrt{a^2+x^2} - a^2 \times$ hyp. Log. $a = a^2 \times$ hyp. Log. $\frac{x+\sqrt{a^2+x^2}}{a} = u$.

* Art. 78.

EXAMPLE XVI.

129. *Let it be proposed to find the Area of the Hyperbola* ABD, *and also the Area of the hyperbolical Sector* CAD; *supposing* C *to be the Center, and* A *the principal Vertex of the Curve.*

Here, putting the Semi-transverse Axis CA $= a$, the Semi-conjugate $= c$, and CB $= x$; we have, by the



Property of the Curve, y ($=$ BD) $= \frac{c}{a}\sqrt{xx-aa}$; and therefore $\dot{u} = y\dot{x} = \frac{c\dot{x}}{a}\sqrt{x^2-a^2} =$ the Fluxion of the Area ABD ‡.

‡ Art. 112.

But to find the Fluxion of the Sector CAD, it is to be observed, that as the said Sector is $=$ CBD $-$ ABD $= \frac{xy}{2} - u$, its Fluxion will therefore be $=$

$\frac{x\dot{y}}{2}+\frac{y\dot{x}}{2}-\dot{u}=\frac{x\dot{y}}{2}-\frac{y\dot{x}}{2}$ (becaufe $\dot{u}=y\dot{x}$ *) which, by fubftituting for y and $\dot{y}$, their Equals $\frac{c}{a}\sqrt{x^2-a^2}$ and $\frac{cx\dot{x}}{a\sqrt{x^2-a^2}}$, is at length reduced to $\frac{ac}{2}\times\frac{\dot{x}}{\sqrt{x^2-a^2}}$: Whereof the Fluent (by *Art.* 126.) is $\frac{ac}{2}\times$ hyp. Log. $x+\sqrt{x^2-a^2}$; which corrected (by making $x=a$) will become $\frac{ac}{2}\times$ hyp. Log. $x+\sqrt{x^2-a^2}-\frac{ac}{2}\times$ hyp. Log. $a=\frac{ac}{2}\times$ hyp. Log. $\frac{x+\sqrt{x^2-a^2}}{a}$ = the Sector ADC: Which, fubtracted from $\frac{cx\sqrt{x^2-a^2}}{2a}$ $(=\frac{\text{BC}\times\text{BD}}{2}=$ the Triangle ABD) leaves $\frac{cx\sqrt{x^2-a^2}}{2a}-\frac{ac}{2}\times$ hyp. Log. $\frac{x+\sqrt{x^2-a^2}}{a}$ for the required Area of the Hyperbola ABD.

* Art. 112.

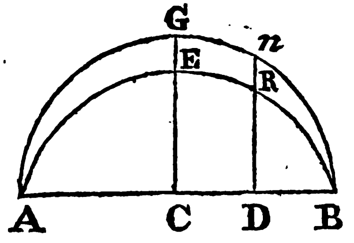
EXAMPLE XVII.

130. *Let the Curve propofed be the Ellipfis* AEB.

Then, putting the tranfverfe Axis AB$=a$, and the Conjugate (2CE) $=c$; we fhall, by the Property of the Curve, have y (DR) $=\frac{c}{a}\sqrt{ax-xx}$, and therefore $\dot{u}$ $(y\dot{x})=\frac{c}{a}\times\dot{x}\sqrt{ax-xx}$ = the Fluxion of the Area ARD.

 But

But $\dot{x}\sqrt{ax-xx}$ is known to expreſs the Fluxion of the correſponding Segment AD*n* of the circumſcribing

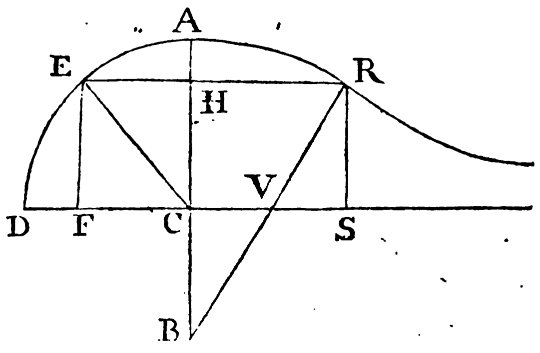


Semi-circle; whoſe Fluent is, therefore, given, by *Art.* 124; which being denoted by A, that of $\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ will, conſequently, be $=\frac{c}{a}\times A$. Hence, the Area of the Segment of an Ellipſis, is to the Area of the correſponding Segment of its circumſcribing Circle, as the leſſer Axis of the Ellipſis is to the greater; whence, it follows that the whole Ellipſis muſt be to the whole Circle in the ſame Ratio.

EXAMPLE XVIII.

131. *Let the Curve* AR &c. *whoſe Area* CARS *you would find, be the Conchoid of* Nicomedes.

Whereof the Equation (putting BC$=a$, and RV (=AC) $=b$) is $x^2y^2=\overline{a+y}^2\times\overline{b^2-y^2}$ (*Vid. Art.* 57.) Which, by Reduction, becomes $x=\frac{a\sqrt{b^2-y^2}}{y}+$



$\sqrt{b^2-y^2}$: But, to bring it down to a, *still*, more simple Form, make $\sqrt{b^2-y^2}$ (=SV) $= z$; then $y = \sqrt{b^2-z^2}$; whence, by Substitution, $x = \frac{az}{\sqrt{b^2-z^2}} + z$; and consequently $\dot{x} =$

$$\frac{a\dot{z}\sqrt{b^2-z^2} + \frac{z\dot{z}}{\sqrt{b^2-z^2}} \times az}{b^2-z^2} + \dot{z} =$$

$$\frac{a\dot{z} \times \overline{b^2-z^2} + az^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z} = \frac{ab^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z};$$

and therefore $\dot{u}\ (y\dot{x}) = \sqrt{b^2-z^2} \times \overline{\frac{ab^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z}} = \frac{ab^2\dot{z}}{b^2-z^2} + \dot{z}\sqrt{b^2-z^2}.$

But now, to exhibit the Fluent hereof; upon C, as a Center, with the Radius AC (b) let a Quadrant of a Circle AED be described, and let RH, produced, meet the Periphery thereof in E, also let EF be parallel to AC, and let CE be drawn: It is evident (because CE (CA) = VR and EF = RS) that CF is also = VS $= z$; and therefore, EF being $(= \sqrt{CE^2 - CF^2}) = \sqrt{b^2-z^2}$, it appears that $\dot{z}\sqrt{b^2-z^2}$ (the second Term

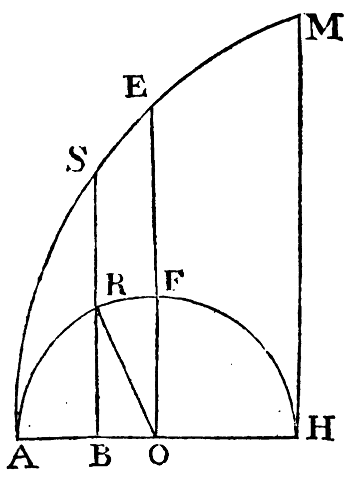
Term of our given Quantity) expresses the Fluxion of the Area AEFC: Whence, if to this Area (found *by the Table of Segments*) the Fluent of the first Term

* Art. 126. $\frac{ab^2\dot{z}}{b^2-z^2}$, or the hyp. Log. of $\frac{b+z}{b-z}$, $\times \frac{1}{2}ab$ *, be added, the Sum will be the whole Area ARCS, that was to be determined.

EXAMPLE XIX.

132. *Let it be required to determine the Area* ASRA *included by the common Cycloid* ASM *and its generating Semi-circle* ARH.

Put the Radius AO (or RO) $= a$, the Sine BR$=y$, the Co-sine OB$=x$, and the Arch AR ($=$RS, by the Property of the Cycloid) $= z$: Then AB being $= a$



$-x$, its Fluxion will be $-\dot{x}$; whence (u) that of the

* Art. 112. Area ARS is $= - z\dot{x}$ *. Now to find the Fluent thereof, make $w = - zx$ ($=$ the Fluent, if z was constant)

ſtant) then $\dot{w}$ being $= -z\dot{x} - x\dot{z}$ *, we ſhall have $\dot{u}$ $(= -z\dot{x}) = \dot{w} + x\dot{z}$. But (*by Art.* 35.) $\dot{z}$ (AR Fluxion) : $\dot{y}$ (BR Fluxion) :: Radius : Co-ſine of the Angle ARB, or its Equal ROB :: OR (a) : OB (x): Therefore, by multiplying Extremes and Means, we get $x\dot{z} = a\dot{y}$: Whence, by Subſtitution $\dot{u}$ $(= \dot{w} + x\dot{z}) = \dot{w} + a\dot{y}$; and conſequently, by taking the Fluent, $u = w + ay = -zx + ay =$ AO × BR — BO × AR = the Area ARS.

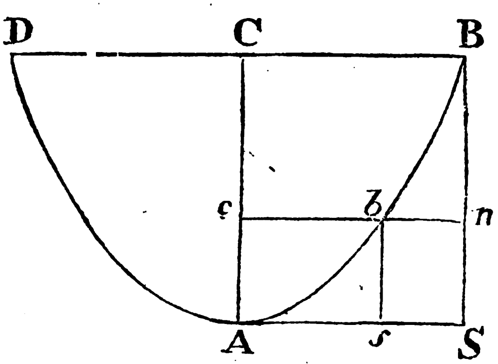
* Art. 10.

Hence it follows that the Area (AEFA) when RB coincides with the Radius FO, is barely = AO × FO = AO²: And that the whole Area AMHFA is truly defined by—ARH ×—OH, or by ARH×OH; that is by four times the Area of the generating Semi-circle.

EXAMPLE XX.

133. *Let the Curve propoſed be the* Catenaria DAB.

Then, drawing BS and *bs* parallel to the Axis AC, and AS and *cbn* perpendicular to the ſame; and making (as uſual) $Ac = x$, $cb = y$ and $Ab = z$, we ſhall have, by



the Property of the Curve, $2ax + x^2 = zz$: Whence $x = \sqrt{a^2 + z^2} - a$, and $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$: From which the

 Value

* Art. 135. Value of $\dot{y}$ (which in all Curves is $= \sqrt{\dot{z}^2 - \dot{x}^2}$ *) will here be found $= \sqrt{\dot{z}^2 - \frac{z^2\dot{z}^2}{a^2+z^2}} = \sqrt{\frac{a^2\dot{z}^2}{a^2+z^2}}$ $\frac{a\dot{z}}{\sqrt{a^2+z^2}}$; and this multiply'd by $\sqrt{a^2+z^2} - a$ ($= bs$) gives $a\dot{z} - \frac{a^2\dot{z}}{\sqrt{a^2+z^2}}$ (= the Rectangle S*b*)

* Art. 112. = the Fluxion of the Area A*sb* *. From whence, by taking the Fluent, the Area itself is found $= az, -a^2$

† Art. 126. $\times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ †: Which therefore deducted from the Rectangle *sc* $(= yx = y\sqrt{a^2+z^2} - ay)$ leaves $y\sqrt{a^2+z^2} - ay - az, + a^2 \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ for the required Area A*bc*. But, since $\dot{y} = \frac{a\dot{z}}{\sqrt{a^2+z^2}}$ we have $y = a \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$; whence, by Substitution, the Area, at last comes out $= y\sqrt{a^2+z^2} - az$, or $= a\sqrt{a^2+z^2} \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$, $- az$.

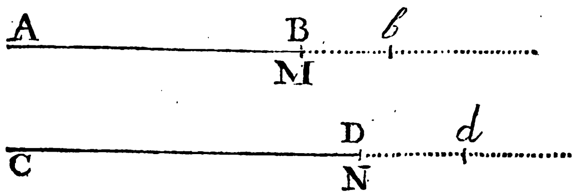
SCHOLIUM.

134. At the Beginning of this, and in the preceding Sections, we have seen how the Fluxions of Quantities are determined, *by conceiving the generating Motion to become uniform at the proposed Position*; according to the true Definition of a Fluxion *: But hitherto no particular Notice has been taken of *the Method of Increments, or indefinitely little Parts*, used (and mistaken) by *many* for *that* of Fluxions: In which the Operations are, for the general Part, exactly the same; and which (tho' less accurate) may be apply'd to good Purpose in finding the Fluxions themselves, in many Cases. For which Reasons it may not be improper to add here a

* Art. 2.

a few Lines on that Head, to ſhew the Beginner how the two Methods differ from each other; eſpecially as we ſhall be enabled, from thence, to draw out ſome Concluſions that will be of Uſe in the enſuing Part of the Work.

It hath been frequently inculcated in the foregoing Pages, that *the Fluxions of Quantities are always meaſured by how much the Quantities themſelves* would be *uniformly augmented in a given Time.* Therefore, if two



Quantities or Lines, AB and CD be generated together, by the uniform (or equable) Motion of two Points B and D, it follows, that any two Spaces B*b* and D*d* *actually* gone over (whereby AB and CD are augmented) in the ſame time, will truly expreſs the Fluxions of the generated Lines AB and CD: Whence it appears that the Increments (or Spaces actually gone over) and the Fluxions are the ſame in this Caſe, where the generating Velocities are equable.

But if, on the contrary, the Velocities of the two Points, in generating the Increments M*b* and N*d*, be ſuppoſed either to increaſe, or to decreaſe, the Lines or Increments ſo generated will, it is plain, no longer expreſs the Fluxions of AB and CD; being greater, or leſs than the Spaces that *might be uniformly* deſcribed, in the ſame Time, with the Velocities at M and N.

If, indeed, thoſe Increments, and the Time of their Deſcription, be taken ſo exceeding ſmall that the Motion of the Points during that Time may be conſidered as equable, the Ratio of the ſaid Increments will then expreſs that of the Fluxions, or be as the Velocity at M to that at N, indefinitely near; but cannot be conceived to be *strictly so*; unless, perhaps, in certain particular Cases.

Hence we see that the *Differential Method*, which proceeds upon these indefinitely little Increments (actually generated) as we do upon Fluxions (or the Spaces that *might be uniformly* generated) differs little, or nothing, from the Method of Fluxions, except in the Manner of Conception, and in Point of Accuracy, wherein it appears defective: And yet it is very certain the Conclusions this Way derived are *mathematically* true; which has afforded Matter of Wonder to *some*: But the Reason why they are so is very easily explained. For, although the *whole complete* Increment is actually understood by the Notation and first Definition (of this Method) yet in the Solution of Problems the exact Measure thereof is not taken, but only that Part of it which would arise from an uniform Increase, agreeable to the Notion of a Fluxion; which admits of a strict Demonstration: But, after all, the *Differential Method* has one Advantage above that of Fluxions, which is, we are not there obliged to introduce the Properties of Motion. Since we reason upon the Increments themselves, and not upon the Manner in which they may be generated.

It has been hinted above, that, though the Increments of Quantities are not, *strictly*, as the Fluxions, yet from them the Ratio of the Fluxions may be deduced; and it appears that the smaller those Increments are taken, the nearer their Ratio will approach to that of the Fluxions. Therefore, if we can, by any Means, find the Ratio to which the said Increments, by conceiving them less and less, do perpetually converge, and which they may approach, before they vanish, nearer than any assignable Difference, that Ratio (called hereafter, for Distinction Sake, *the Ratio limiting that of the Increments*) will be, *strictly*, that of the Fluxions.

This will more particularly appear from the following Instances; wherein the Manner of deriving the Ratio of the Fluxions, from that of the Increments, is shewn.

1°.

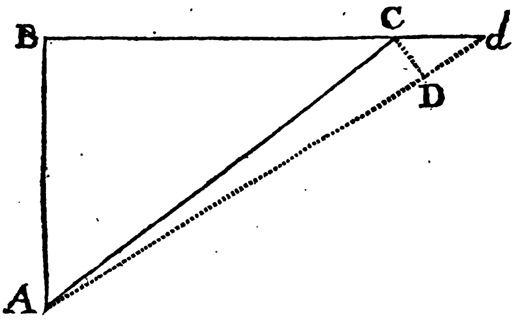
1°. *Let it be propofed to determine the Ratio of the Fluxions of x and x^2.*

Now, if x be fuppofed to be augmented by any (fmall) Quantity $\acute{x}$, fo as to become $x+\acute{x}$, its Square (x^2) will be augmented to $\overline{x+\acute{x}}|^2 = x^2 + 2x\acute{x} + \acute{x}\acute{x}$; whence the Increment of x^2 will be $2x\acute{x}+\acute{x}\acute{x}$; which therefore is to ($\acute{x}$) the Increment of x, as $2x+\acute{x}$ to 1. Hence, becaufe the leffer $\acute{x}$ is taken, the nearer this Ratio approaches to that of $2x$ to 1, which is its *Limit*, the Ratio of the Fluxions will therefore be expreffed by that of $2x$ to 1, or, which is the fame, by that of $2x\dot{x}$ to $\dot{x}$ (*as in Art.* 6.)

2°. *Let the Ratio of the Fluxions of x and x^n be required.*

Then, if x be augmented to $x+\acute{x}$, x^n will be augmented to $\overline{x+\acute{x}}|^n = x^n + nx^{n-1}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x}^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}^3$ &c. (*Vid. Art.* 99.) Whence the Increments of x and x^n will be to each other as 1 to $nx^{n-1} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}\acute{x}$ &c. Where the fmaller $\acute{x}$ is taken, the nearer the Ratio will approach to that of of 1 to nx^{n-1}; which appears to be its Limit: Therefore this laſt Ratio, or that of $\dot{x}$ to $nx^{n-1}\dot{x}$, is the Ratio of the Fluxions required. *(Vid. Art.* 8.)

3°. *Let it be propoſed to determine the Proportion of the Fluxions of the Sides* AC *and* BC, *of a right-angled, plane Triangle* ABC; *ſuppoſing the Perpendicular* AB *to remain invariable.*



If C*d* be aſſumed to repreſent any Increment of BC; and D*d*, the correſponding Increment of AC (=AD) the Ratio of thoſe Increments will be, univerſally, expreſſed by that of the Sine of the Angle CD*d* to the Sine of the Angle DC*d* (*by plane Trigonometry*) and the leſs the Increments are ſuppoſed to be, the nearer will the Angle CD*d* approach to a right one, or to an Equality with B; which is its Limit: And the nearer will DC*d* approach, at the ſame time, to an Equality with BAC. Therefore *the Ratio here limiting that of the Increments* is that of the Sine of B (or Radius) to the Sine of BAC: Which alſo expreſſes that of the required Fluxions. *(Vid. Art.* 35.)

In the ſame way the Proportion of the Fluxions of other Kinds of algebraical and geometrical Quantities may

may be inveſtigated; but it will be unneceſſary to dwell longer upon this Head: I ſhall therefore only add one other Obſervation from hence (which will be of uſe hereafter) relating to the Value of an algebraic Fraction, in that particular Circumſtance when both its Numerator and Denominator become equal to Nothing, or vaniſh, at the ſame time. Which Value (it follows from above) will *be found by dividing the Fluxion of the Numerator by that of the Denominator.*

For, ſince the Value of any Fraction, in that Circumſtance, is to be looked on as *the limiting Ratio* towards which its two Terms converge, before they vaniſh, and ſeeing the Fluxions are, always, expreſſed by that Ratio, the Truth of the Rule, or Poſition, is manifeſt.

An Example, however, may not be improper:

Let therefore the Fraction $\frac{x^2-a^2}{x-a}$ be propounded, to find the Value thereof when $x=a$. In which Caſe, the true Value ſought, or the Fluxion of the Numerator divided by that of the Denominator, is $=\frac{2x\dot{x}}{\dot{x}}$ $=2x=2a$. And that this is the true Value, may be confirmed by common Diviſion, whereby the Fraction propoſed is reduced to $x+a$; whoſe Value when $x=a$, is therefore $=2a$, *the very ſame as before.*

SEC-

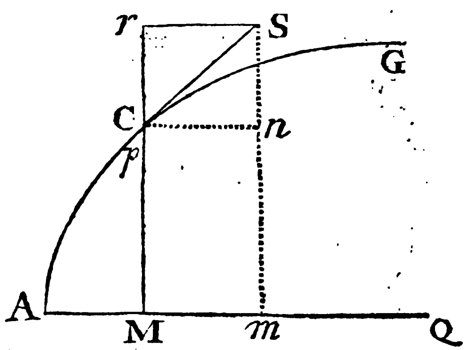
SECTION VIII.

The Use of Fluxions in the Rectification, or finding the Lengths, of Curves.

CASE I.

135. LET ACG *be a Curve of any Kind whose Ordinates are parallel to themselves and perpendicular to the Axis* AQ.

If the Fluxion of the Absciss AM be denoted by M*m*, or by C*n* (equal and parallel to M*m*) and *n*S,



equal and parallel to C*r*, be taken to represent the corresponding Fluxion of the Ordinate MC; then will the Diagonal CS (touching the Curve in C *) be the Line which the generating Point (*p*) would describe, was its Motion to become uniform at C (*Vid. Art.* 48 and 49.) which Line, therefore, truly expresses the Fluxion of the Space AC gone over, according to *the Definition* †.

* Art. 48 and 49.

† Art. 2.

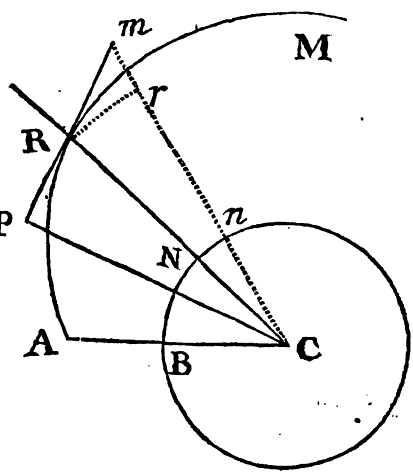
Hence, putting AM$=x$, CM$=y$, and AC$=z$, we have $\dot{z}$ ($=$CS$=\sqrt{Cn^2+Sn^2}$) $=\sqrt{\dot{x}^2+\dot{y}^2}$; from which, and the Equation of the Curve, the Value of z may be determined.

CASE

CASE II.

136. *Let all the Ordinates of the proposed Curve* ARM *be referred to a Center* C.

Then, putting the Tangent RP (intercepted by the Perpendicular CP) $= t$, the Arch BN, of a Circle described about the Center C$=x$; the Radius CN (or CB) $=a$, &c. (*Vid. Art.* 113) we have $\dot{z} : \dot{y} :: y$ (CR)



$: t$ (RP *); and consequently $\dot{z} = \frac{y\dot{y}}{t}$: From whence * Art. 35. the Value of z will be found, if the Relation of y and t is given.

But in other Cases it will be better to work from the following Equation, *viz.* $\dot{z} = \sqrt{\dot{y}^2 + \frac{y^2 \dot{x}^2}{a^2}}$. Which is thus derived.

Let the Right Line, CR, be conceived to revolve about the Center C; then since the Celerity of the generating

nerating Point R in a Direction perpendicular to CR is to $(\dot{x})$ the Celerity of the Point N, as CR (y) to CN (a) it will therefore be truly represented by $\frac{y\dot{x}}{a}$: Which being to $(\dot{y})$ the Celerity in the Direction of CR, produced, as CP (s) : RP (t) * it follows that $\frac{y^2\dot{x}^2}{a^2} : \dot{y}^2 :: s^2 : t^2$: Whence, by Composition, $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 : \dot{y}^2 :: s^2 + t^2 \; (y^2) : t^2$; therefore $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 = \frac{y^2\dot{y}^2}{t^2}$, and consequently $\sqrt{\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2} \left(= \frac{y\dot{y}}{t}\right) = \dot{z}$; as was to be shewn.

* Art. 35.

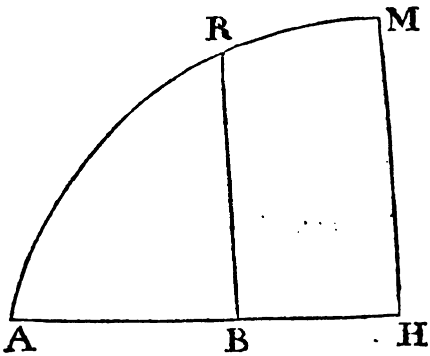
But the same Conclusion may be more easily deduced from the Increments of the flowing Quantities, according to the preceding Scholium.

For, if R*m*, *rm* and N*n* be assumed to represent $(\acute{z}, \acute{y}$ and $\acute{x})$ any very small corresponding Increments of AR, CR and BN, it will be as CN (a) : CR (y) :: $\acute{x}$ (the Arch N*n*) : the similar Arch R*r* $= \frac{y\acute{x}}{a}$. And, if the Triangle R*rm* (which, while the Point *m* is returning back to R, approaches continually nearer and nearer to a Similitude with CRP) be considered as *rectilineal*, we shall also obtain $\acute{z}^2$ $(= Rm^2 = Rr^2 + rm^2)$ $= \frac{y^2\acute{x}^2}{a^2} + \acute{y}^2$: Whence, by writing $\dot{z}$, $\dot{x}$ and $\dot{y}$ for $\acute{z}$, $\acute{x}$ and $\acute{y}$ (according to the Scholium) there comes out $\dot{z}^2 = \frac{y^2\dot{x}^2}{a^2} + \dot{y}^2$, *as before.*

EXAMPLE I.

137. *Let the Curve* ARM *whose Length is sought, be the Semi-cubical Parabola.*

Whereof the Equation being $ax^2 = y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, we thence have $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: Whence $\dot{z}$ $(=\sqrt{\dot{y}^2 + \dot{x}^2}$ *) * Art. 135.



$= \sqrt{\dot{y}^2 + \frac{9y\dot{y}^2}{4a}} = \frac{\dot{y} \times \overline{4a + 9y}^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$. Whose Fluent (found by the common Rule) is $\frac{\overline{4a + 9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$; which, corrected (by making $y = 0$) becomes $\frac{\overline{4a + 9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27} = z$.

EX-

EXAMPLE II.

138. *Let the Curve proposed be a Parabola of any (other) Kind.*

Then $x = \frac{y^n}{a^{n-1}}$ being a general Equation to all Kinds of Parabolas, we here have $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$, and therefore $\dot{z}\ (= \sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\dot{y}^2 + \frac{n^2 y^{2n-2} \dot{y}^2}{a^{2n-2}}} = \dot{y} \times \overline{1 + \frac{n^2 y^{2n-2}}{a^{n-2}}}\Big|^{\frac{1}{2}}$: Whose Fluent, universally expressed in an Infinite Series, is $y + \frac{n^2 y^{2n-1}}{\overline{2n-1} \times 2a^{2n-2}} - \frac{n^4 y^{4n-3}}{\overline{4n-3} \times 8a^{4n-4}} + \frac{n^6 y^{6n-5}}{\overline{6n-5} \times 16a^{6n-6}}$ &c. $= z$.

But, when $2n-2$, the Index of y, in the given Fluxion, is either equal to Unity, or to any *aliquot Part* of it, the Fluent may be accurately had in finite Terms, *by Article* 84.

For, by putting $\frac{1}{2n-2} = v$, and $\frac{n^2}{a^{2n-2}} = c$, our Fluxion $\left(\overline{1 + \frac{n^2 y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y}\right)$ is, in the first place, reduced to $\overline{1 + cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}} \times \dot{y}$: Which being compared with

with $\overline{a+cz^{n}}|^{m} \times dz^{rn-1}\dot{z}$, the general Expression in the foresaid *Article*, we have $a = 1$; $z = y$, $n = \frac{1}{v}$, $m = \frac{1}{2}$, $d = 1$, $\dot{z} = \dot{y}$, $rn - 1 = 0$, or $\frac{r}{v} - 1 = 0$; whence $r = v$, $s\ (r + m) = v + \frac{1}{2}$; and consequently

$$\frac{d \times \overline{a+cz^{n}}|^{m+1}}{snc} \times \frac{z^{rn-n}}{1} - \frac{\overline{r-1} \times az^{rn-2n}}{\overline{s-1} \times c} + \&c. = ^{*}$$

$$\frac{\overline{1+cy^{\frac{1}{v}}}|^{\frac{3}{2}}}{c+\frac{c}{2v}} \times y^{\frac{v-1}{v}} - \frac{\overline{v-1} \times y^{\frac{v-2}{v}}}{\overline{v-\frac{1}{2}} \times c} + \frac{\overline{v-1} \times \overline{v-2} \times y^{\frac{v-3}{v}}}{\overline{v-\frac{1}{2}} \times \overline{v-\frac{3}{2}} \times c^{2}} - \&c. = \text{the Fluent of}$$

$\overline{1+cy^{\frac{1}{v}}}|^{\frac{1}{2}} \times \dot{y}$; which was to be determined, and which will (it is plain) always terminate in v Terms, when v, or its Equal $\frac{1}{2n-2}$, is a whole positive Number.

If $\frac{2v+1}{2v}$ (derived from $v = \frac{1}{2n-2}$) be substituted for its Equal n, the Equation of the Curve, will be changed to $ax^{2v} = y^{2v+1}$; which, if v be expounded by 1, 2, 3, 4 &c. successively, will become $ax^{2} = y^{3}$, $ax^{4} = y^{5}$, $ax^{6} = y^{7}$, $ax^{8} = y^{9}$ &c. respectively: In all which Cases the Length of the Curve may therefore be *accurately* had from the Fluent above exhibited.

* Art. 84.

Moreover, if n be assumed $= 2$ (or $v = \frac{1}{2}$) the general Equation, $x = \frac{y^n}{a^{n-1}}$, will then become $x = \frac{y^2}{a}$; answering to the common (or conical) Parabola.

And therefore in that Case $\dot{z}$ $\left(= \overline{1 + \frac{n^2 y^{2n-2}}{a^{2n-2}}}\right|^{\frac{1}{2}} \times \dot{y})$ is $= \dot{y}\sqrt{1 + \frac{4y^2}{a^2}} = \frac{\dot{y}\sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a} = \frac{\dot{y}\sqrt{b^2 + y^2}}{b}$ (by putting $b = \frac{1}{2}a$) $= \frac{\dot{y} \times \overline{b^2 + y^2}}{b\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2\dot{y} + y^2\dot{y}}{\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2 y\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2\dot{y}}{\sqrt{b^2 + y^2}}$: Where, the Fluent of the first Term (of the Fluxion so transformed) being $= \frac{1}{2}\sqrt{b^2y^2 + y^4}$ (or $\frac{1}{2}y\sqrt{b^2 + y^2}$) by the common Rule; and that of the second Term $= \frac{1}{2}b^2 \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$, * it follows that the Length of the Curve will, in this Case, be $=$

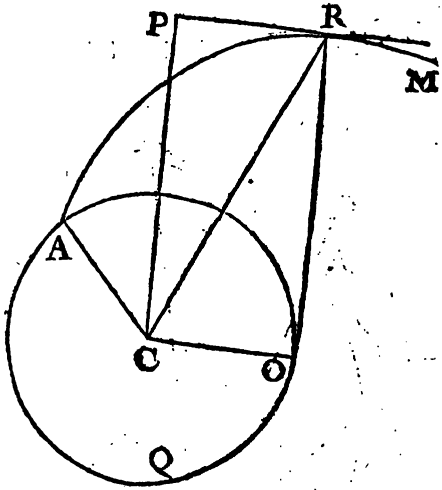
$$\frac{\frac{1}{2}y\sqrt{b^2 + y^2}}{b} + \frac{1}{2}b \times \text{hyp. Log.}\ \frac{y + \sqrt{b^2 + y^2}}{b}.$$

* Art. 126.

EX.

EXAMPLE III.

139. *Let the Curve proposed, be the Involute of a Circle;* whose Nature is such, that the Part PR of the Tangent intercepted by the Point of Contact and the Perpendicular CP, is every where equal to the Radius CO of the generating Circle:



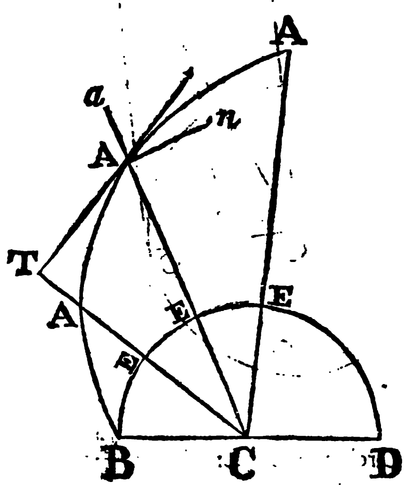
Therefore $\dot{x}$ $\left(=\frac{y\dot{y}}{t}\ *\right)$ being here $=\frac{y\dot{y}}{a}$, we first get $x=\frac{y^2}{2a}$; which corrected, by making $y=a$ $(=AC)$ becomes $\frac{y^2-a^2}{2a}$ $\left(\frac{CP^2}{2CA}\right)$ the true Measure of the required Arch AR.

* Art. 136.

EXAMPLE IV.

140. *In which the Spiral of* Archimedes *is propofed.*

Where, the Value of t (AT) being denoted by $\frac{by}{\sqrt{b^2+y^2}}$ (*Vid. Art.* 62.) we get $\dot{x}$ $\left(=\frac{y\dot{y}}{t}\right)$ $=\frac{\dot{y}\sqrt{b^2+y^2}}{b}$: Which Fluxion being exactly the



fame as that expreffing the Arch of the common Parabola, found in *Article* 138. its Fluent will therefore be truly reprefented by the Meafure of the faid Arch, or by $\frac{\frac{1}{2}y\sqrt{b^2+y^2}}{b}+\frac{1}{2}b\times hyp.\ Log.\ \frac{y+\sqrt{b^2+y^2}}{b}$, the Value there exhibited.

EXAMPLE V.

141. *Let the Curve be a Spiral whose Equation is* $a^{m-1}x = y^m$ (*Vid. Art.* 136.)

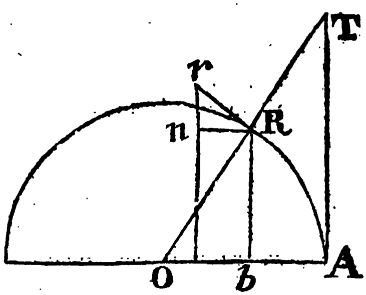
In which Case $\dot{x}$ being $= \frac{m y^{m-1}\dot{y}}{a^{m-1}}$, it is evident that $\dot{z}$ $\left(= \sqrt{\dot{y}^2 + \frac{y^2 \dot{x}^2}{a^2}}\ ^*\right) = \sqrt{\dot{y}^2 + \frac{m^2 y^{2m} \dot{y}^2}{a^{2m}}}$ $= \dot{y}\sqrt{1 + \frac{m^2 y^{2m}}{a^{2m}}}$; and therefore $z = y + \frac{m^2 y^{2m+1}}{\overline{2m+1} \times 2a^{2m}}$ $- \frac{m^4 y^{4m+1}}{\overline{4m+1} \times 8a^{4m}} + \frac{m^6 y^{6m+1}}{\overline{6m+1} \times 16a^{6m}}$ &c. Which Value may be otherwise had, without an Infinite Series, when $\frac{1}{2m}$ is a whole positive Number, *Vide Art.* 138.

* Art. 136.

EXAMPLE VI.

142. *Where, the Right-sine, Versed-sine, Tangent, or Secant of an Arch of a Circle, being given, 'tis required to find the Length of the Arch itself in Terms thereof.*

Put the Versed-sine $Ab = x$, the Right-sine $Rb = y$, the Tangent $AT = t$, the Secant $OT = s$, the Arch $AR = z$, and the Radius AO, or RO, $= a$; also let $Rn = \dot{x}$, $nr = \dot{y}$ and $Rr = \dot{z}$: Since the Angle rnR ($=$ Right-angle) $= ObR$, and rRn ($=$ Right-angle $- nRO$) $= ORb$, the Triangles rRn and ORb



 are

are therefore equi-angular; and it will be, R*b* (y) : OR (a) :: R*n* ($\dot{x}$) : R*r* ($\dot{z}$) $= \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ (because, by the Property of the Circle $\sqrt{2ax - xx} = y$.) Also, O*b* ($\sqrt{a^2 - y^2}$) : OR (a) :: *nr* ($\dot{y}$) R*r* ($\dot{z}$) $= \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$. These two Values exhibit the Fluxion of the Arch in Terms of the Versed-sine and Right-sine respectively: But, to get the same, in Terms of the Tangent and Secant, we have (*by sim. Triangles*) OT ($= s = \sqrt{a^2 + t^2}$) : OA (a) :: OR (a) : Ob $= \frac{a^2}{s} = \frac{a^2}{\sqrt{a^2+t^2}}$: Hence A$b = a - \frac{a^2}{s} = a - \frac{a^2}{\sqrt{a^2+t^2}}$; whose Fluxion is therefore $= \frac{a^2\dot{s}}{s^2} = \frac{a^2t\dot{t}}{\overline{a^2+t^2}|^{\frac{3}{2}}}$: Whence (again by similar Triangles) AT ($= \sqrt{s^2 - a^2} = t$) : OT ($= s = \sqrt{a^2+t^2}$) :: R*n* : R*r* $= \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}} = \frac{a^2\dot{t}}{a^2 + t^2} = \dot{z}$.

Now, from any one of the four Forms of Fluxions $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}, \frac{a\dot{y}}{\sqrt{a^2 - y^2}}, \frac{a^2\dot{t}}{a^2 + t^2}, \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}\right)$ here found, the Value of the Arch itself (by taking the Fluent, in an Infinite Series) will likewise become known.

But, the third Form, expressed in Terms of the Tangent, being intirely free from radical Quantities, will be the most ready in Practice, especially where the required Arch is but small; though the Series arising from the first Form, always, converges the fastest.

If,

If, therefore, $\frac{a^2\dot{t}}{a^2+t^2}$ be now converted to an Infinite Series, we shall have $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$ &c. and consequently $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ &c. $=$ AR. Where, if (*for Example Sake*) AR be supposed an Arch of 30 Degrees, and AO (to render the Operation more easy) be put $=$ Unity, we shall have $t = \sqrt{\frac{1}{3}} = .5773502$ (because Ob $\sqrt{\frac{3}{4}}$: bR $(\frac{1}{2})$ :: OA (1) : AT $(t) = \sqrt{\frac{1}{3}}$)

Whence

$t^3 \; (= t \times t^2 = t \times \frac{1}{3}) = .1924500$

$t^5 \left(= t^3 \times t^2 = \frac{t^3}{3}\right) = .0641500$

$t^7 \left(= t^5 \times t^2 = \frac{t^5}{3}\right) = .0213833$

$t^9 \left(= t^7 \times t^2 = \frac{t^7}{3}\right) = .0071277$

$t^{11} \left(= t^9 \times t^2 = \frac{t^9}{3}\right) = .0023759$

$t^{13} \left(= t^{11} \times t^2 = \frac{t^{11}}{3}\right) = .0007919$

$t^{15} \left(= t^{13} \times t^2 = \frac{t^{13}}{3}\right) = .0002639$

&c.

And therefore AR $= .5773502 - \frac{.1924500}{3} + \frac{.0641500}{5} - \frac{.0213833}{7} + \frac{.0071277}{9} - \frac{.0023759}{11} +$

$$+\frac{.0007919}{13}-\frac{.0002639}{15}+\frac{.0000879}{17}-\frac{.0000293}{19}$$

$$+\frac{.0000097}{21}-\frac{.0000032}{23}=.5235987$$: Which multiply'd by 6 gives 3.141592 + for the Length of the Semi-periphery of the Circle whose Radius is Unity.

At *Article* 126. certain Forms of Fluxions were pointed out, whose Fluents are explicable by means of hyperbolical Spaces, or a *Table of Logarithms*: Which Forms, it is observable, agree in every thing, but the Signs (and constant Quantities) with those exhibited above, for the Arch of a Circle. And these last, like them, may serve as so many (other) Theorems for finding Fluents by means of a *Table of Sines, Tangents and Secants.* But, as such a Table is usually calculated to a Radius of 1,000000 *&c.* (or Unity) the following Equations, derived from those above, being adapted to that Radius, will be rather more commodious.

$$\text{Thus, the Fluent of}\left\{\begin{array}{c}\dfrac{\dot{w}}{\sqrt{2aw-w^2}}\\[2ex] \dfrac{\dot{w}}{\sqrt{a^2-w^2}}\\[2ex] \dfrac{a\dot{w}}{a^2+w^2}\\[2ex] \dfrac{a\dot{w}}{w\sqrt{w^2-a^2}}\end{array}\right\}\text{is equal to the Arch whose}\left\{\begin{array}{l}\text{Versed-sine}\\[2ex] \text{Right-sine}\\[2ex] \text{Tangent}\\[2ex] \text{Secant}\end{array}\right\}\text{is } \frac{w}{a}\text{; and Radius Unity.}$$

The way of deducing these Expressions, from the foregoing ones, is extremely easy: For, if A be put to denote the Arch whose Radius is Unity, and whose Versed sine, Right-sine, Tangent, or Secant is $\frac{w}{a}$ (according to the different Cases here specified.) Then, because similar Arcs, of unequal Circles, are as their Radii,

Radii, it will be $1 : a :: A : (aA)$ the Length of the Arch AR (*see the Figure.*) Therefore, the Fluent of $\frac{a\dot{x}}{\sqrt{2ax - xx}}$ (or $\frac{a\dot{w}}{\sqrt{2aw - w^2}}$, putting $w = x$) being $= aA$ (AR), that of $\frac{\dot{w}}{\sqrt{2aw - w^2}}$ must necessarily be $= A$: And in the very same Manner the other Forms are made out.

EXAMPLE VII.

143. *Let the proposed Curve be the common Cycloid.*

Then, if the Radius AO of the generating Semi-circle* be denoted by a, we shall have BR $= \sqrt{2ax - x^2}$; and the Fluxion thereof $= \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}}$: Which being added to $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}\right)$ the Fluxion of AR or its Equal RS (given by the preceding Article) we thence get $\frac{2a\dot{x} - x\dot{x}}{\sqrt{2a\dot{x} - x^2}} = \frac{\dot{x} \times \overline{2a - x}}{x^{\frac{1}{2}} \times \overline{2a - x}|^{\frac{1}{2}}} = \frac{\dot{x}}{x^{\frac{1}{2}}} \times \overline{2a - x}|^{\frac{1}{2}}$, for the true Fluxion of the Ordinate BS of the Cycloid.

* See Fig. Art. 132.

Hence $\dot{z}\left(\sqrt{\dot{x}^2 + \dot{y}^2}\ ^*\right) = \sqrt{\dot{x}^2 + \frac{\dot{x}^2 \times \overline{2a - x}}{x}} = \dot{x}\sqrt{\frac{2a}{x}} = \overline{2a}|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$; and consequently, by taking the Fluent, $z = \overline{2a}|^{\frac{1}{2}} \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{2ax} =$ the Arch AS of the Cycloid.

* Art. 135.

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EXAMPLE VIII.

144. *Wherein it is required to determine the Length of the Arch of the common Hyperbola.*

In this Case (the Semi-transverse Axis being represented by b, and the Semi-conjugate by c) we have $\frac{b^2y^2}{c^2} = 2bx + x^2$; and therefore $x = \frac{b\sqrt{c^2+y^2}}{c}$ $- b$: Hence $\dot{x} = \frac{by\dot{y}}{c\sqrt{c^2+y^2}}$, and $\dot{z}$ $(= \sqrt{\dot{y}^2+\dot{x}^2})$ $\sqrt{\dot{y}^2 + \frac{b^2y^2\dot{y}^2}{c^2 \times \overline{c^2+y^2}}} = \dot{y}\sqrt{1+\frac{b^2y^2}{c^4+c^2y^2}}$; which, by converting $\frac{b^2y^2}{c^4+c^2y^2}$ into an *Infinite Series*, becomes $\dot{y}\sqrt{1+\frac{b^2y^2}{c^4}-\frac{b^2y^4}{c^6}+\frac{b^2y^6}{c^8}-\frac{b^2y^8}{c^{10}}}$ &c. But still we have the Square Root to extract; In order thereto let it be assumed $= 1 + Ay^2 + By^4 + Cy^6 + Dy^8$ &c. Then, by squaring, and transposing (*Vid. Art.* 98.) there arises

$$\left.\begin{array}{l} 1+2Ay^2+2By^4+2Cy^6+2Dy^8 \text{ \&c.} \\ \qquad +A^2y^4+2ABy^6+2ACy^8 \text{ \&c.} \\ \qquad\qquad +B^2y^8 \text{ \&c.} \\ -1-\frac{b^2}{c^4}\times y^2+\frac{b^2}{c^6}\times y^4-\frac{b^2}{c^8}\times y^6+\frac{b^2}{c^{10}}\times y^8 \text{ \&c.} \end{array}\right\} = 0$$

Hence $A = \frac{b^2}{2c^4}$; $B = -\frac{b^2}{2c^6} - \frac{1}{2}A^2 = -\frac{b^2}{2c^6}$ $-\frac{b^4}{8c^8}$; $C = \frac{b^2}{2c^8} - AB = \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}$, &c. &c. Therefore $\dot{z}$ $(= \dot{y}\sqrt{1+\frac{b^2y^2}{c^4} \text{ \&c.}} = \dot{y} \times$ $\overline{1+Ay^2+By^4 \text{ \&c.}}) = \dot{y} + \frac{b^2}{2c^4} \times y^2\dot{y} - \overline{\frac{b^2}{2c^6}+\frac{b^4}{8c^8}} \times$

$y^4\dot{y}$

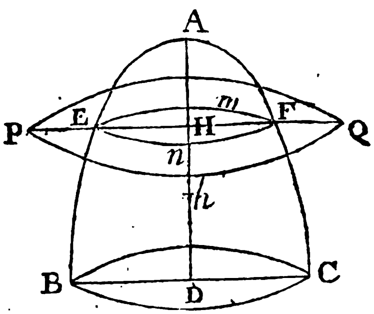
$y^4\dot{y} + \overline{\frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}} \times y^6\dot{y}$ &c. And consequently $\dot{z} = y + \frac{b^2y^3}{6c^4} - \overline{\frac{b^2}{c^2} + \frac{b^4}{4c^4}} \times \frac{y^5}{10c^4} + \overline{\frac{b^2}{c^2} + \frac{b^4}{2c^4} + \frac{b^6}{8c^6}} \times \frac{y^7}{14c^6}$ &c.

By the very ſame way of proceeding the Arch of an Ellipſis may be found, the Equations of the two Curves differing in nothing but their Signs.

SECTION IX.

The Application of FLUXIONS *in inveſtigating the Contents of Solids.*

145. LET ABC repreſent any Solid; conceived to be generated (or deſcribed) by a Plane PQ paſſing over it, with a parallel Motion: Let H*b* (perpendicular to PQ) be taken to expreſs the Fluxion of AH (*x*) or the Velocity with which the generating Plane is carry'd; alſo let the Area of the Part, E*m*F*n*, of the Plane intercepted by, or contained in, the Solid, be denoted by *A*: Then it follows, from *Art.* 2 and 5. that the Fluxion of the Solid AEF, will be expreſſed by $A\dot{x}$. From whence, by



expounding *A* in Terms of *x*, (according to the Nature of the Figure) and then taking the Fluent, the Content of of the Solid (which we shall, always, hereafter represent by s) will be given.

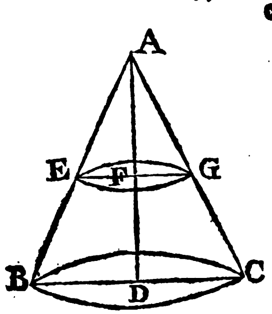
But, when the proposed Solid is that arising from the Revolution of any given Curve AEB about AHD, as an Axis, the Fluxion ($\dot{s}$) of the Solidity may be exhibited in a Manner more convenient for Practice: For, putting the Area (3,141592 &c.*) of the Circle, whose Radius is Unity, $=p$, and the Ordinate EH $=y$, it will be $1^2 : y^2 :: p : (py^2)$ the Area of the Circle $EmFn$, which being wrote above instead of A, we have $\dot{s} = py^2\dot{x}$. The Use of which will be sufficiently shewn in the following Examples.

* Art. 124.

EXAMPLE I.

146. *Let it be proposed to find the Content of a Cone* ABC.

Put the given Altitude (AD) of the Cone $=a$, and the Semi-diameter (BD) of its Base $=b$: Then, the Distance (AF) of the Circle EG, from the Vertex A, being denoted by x, &c. we have, by similar Triangles, as $a : b :: x : \text{EF}\ (y) = \frac{bx}{a}$. Whence, in this Case, $\dot{s}$



$(= py^2\dot{x}) = \frac{pb^2x^2\dot{x}}{a^2}$; and consequently $s = \frac{pb^2x^3}{3a^2}$; which, when $x=a$ (= AD) gives $\frac{pb^2a}{3}$ $(=p \times \text{BD}^2 \times \frac{1}{3}\text{AD})$ for the Content of the whole Cone ABC. Which appears, from hence, to be just $\frac{1}{3}$ of a Cylinder of the same Base and Altitude.

EXAMPLE II.

147. *Where, let the Solid proposed be a parabolic Conoid, or that arising from the Revolution of any Kind of Parabola about its Axis.*

Then, from the Equation $a^{m-n}x^n = y^m$, of the generating Curve, we get $y = a^{\frac{m-n}{m}} \times x^{\frac{n}{m}}$, and $\dot{s}$ $(= py^2\dot{x})$ $= pa^{\frac{2m-2n}{m}} \times \dot{x}x^{\frac{2n}{m}}$; and therefore $s = pa^{\frac{2m-2n}{m}} \times \frac{x^{\frac{2n}{m}+1}}{\frac{2n}{m}+1} = pa^{\frac{2m-2n}{m}} \times \frac{mx^{\frac{2n}{m}+1}}{2n+m} = pa^{\frac{2m-2n}{m}} \times x^{\frac{2n}{m}} \times \frac{mx}{2n+m} = py^2 \times \frac{mx}{2n+m} =$ the Content of the Solid; which therefore is to (py^2x) the Content of the circumscribing Cylinder, as m to $2n+m$. Whence the Solid generated by the conical Parabola (where $m = 2$, and $n = 1$) appears to be just $\frac{1}{2}$ of its circumscribing Cylinder.

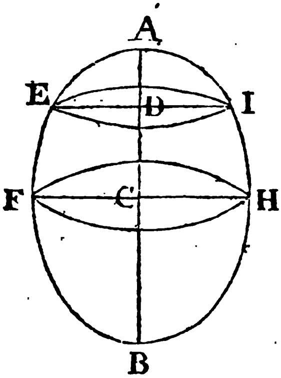
EXAMPLE III.

148. *Let the proposed Solid* AFBH *be a Spheroid.*

In which Case, putting the Axis AB, about which the Solid is generated, $= a$, and the other Axis FH, of the generating Ellipsis $= b$, it follows, from the Property of the Ellipsis, that $a^2 : b^2 :: x \times \overline{a-x}$ (AD × BD) $: y^2$ (DE²) $= \frac{b^2}{a^2} \times \overline{ax-xx}$: Whence we have $\dot{s}$ $(= py^2\dot{x}$ *$) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} - x^2\dot{x}}$; and * Art. 145.

$s = \frac{pb^2}{a^2} \times \overline{\frac{1}{2}axx - \frac{1}{3}x^3} =$ the Segment AIE. Which,

when



when AD $(x) = AB (a)$, becomes $(\frac{pb^2}{a^2} \times \overline{\frac{1}{2}a^3 - \frac{1}{3}a^3})$ $\frac{1}{6}pab^2$ = the Content of the whole Spheroid. Where, if b (FH) be taken $= a$ (AB) we shall also get $\frac{1}{6}pa^3$ for the true Content of the Sphere whose Diameter is a. Hence a Sphere, or a Spheroid, is $\frac{2}{3}$ of its circumscribing Cylinder; for the Area of the Circle FH being expressed by $\frac{pb^2}{4}$, the Content of the Cylinder whose Diameter is FH, and Altitude AB, will therefore be $\frac{pb^2a}{4}$; of which $\frac{1}{6}pab^2$, is, evidently, two third Parts.

EXAMPLE IV.

149. *Let the Solid, whose Content you would find, be the hyperbolical Conoid.*

Then, from the Equation, $y^2 = \frac{b^2}{a^2} \times \overline{ax + xx}$, of the generating Hyperbola, we have $\dot{s}$ $(py^2\dot{x}) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} + x^2\dot{x}}$, and consequently $s = \frac{pb^2}{a^2} \times \overline{\frac{1}{2}ax^2 + \frac{1}{3}x^3}$ = the Content of the Conoid; which therefore is to $(\frac{pb^2}{a^2} \times \overline{ax + x^2} \times x)$ that of a Cylinder of the same Base and Altitude, as $\frac{1}{2}a + \frac{1}{3}x$ to $a + x$. This Ratio, if x be extremely small, will become as 1 to 2 very nearly: Whence it may be inferr'd, that the Content of of a very small Part of any Solid, generated by a Curve, whose Ray of Curvature at the Vertex is a finite Quantity, is half that of a Cylinder of the same Base and Altitude, very nearly: Because any such Curve, for a small Distance, will differ insensibly from an Hyperbola, whose Radius of Curvature, at the Vertex, is the same.

This might have been inferred, either, from the common parabolic Conoid, or the Spheroid, in the preceding Examples; but other Observations would not allow Room for it there.

EXAMPLE V.

150. *In which the proposed Solid is that arising from the Rotation of the Cissoid of* Diocles, *about its Axis.*

Here, y^2 being $= \frac{x^3}{a-x}$, * we have $\dot{s}\ (py^2\dot{x}) =$ * Art. 56.

$\frac{px^3\dot{x}}{a-x}$. But, in Cases like this, (where the Denominator is rational and the variable Quantity in the Numerator of several Dimensions, it will be necessary to divide the latter by the former, in order to obtain the Fluent, by lessening the Number of Dimensions: Thus, dividing $px^3\dot{x}$ by $-x+a$, according to the Manner of compound Quantities, the Work will stand thus:

$$
\begin{array}{l}
-x+a)\ px^3\dot{x}-0 \quad (-px^2\dot{x}-pax\dot{x}-pa^2\dot{x} \\
\qquad\quad px^3\dot{x}-pax^2\dot{x} \\
\hline
\qquad\qquad +pax^2\dot{x}-0 \\
\qquad\qquad +pax^2\dot{x}-pa^2x\dot{x} \\
\hline
\qquad\qquad\qquad +pa^2x\dot{x}-0 \\
\qquad\qquad\qquad +pa^2x\dot{x}-pa^3\dot{x} \\
\hline
\qquad\qquad\qquad\qquad +pa^3\dot{x} \\
\hline
\end{array}
$$

Where, the Quotient being $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}$, and the Remainder $pa^3\dot{x}$, the Value of the given Fraction $\frac{px^3\dot{x}}{a-x}$,

will

will therefore be truly expressed by $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}+\frac{pa^3\dot{x}}{a-x}$: Whose Fluent, properly corrected, is $-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times$ hyp. Log. $\frac{a}{a-x}$ Vid. Art. 126.

EXAMPLE VI.

151. *Let the Solid be that arising from the Rotation of the Conchoid of* Nicomedes *about its Axis.*

The Sub-tangent $\frac{y\dot{x}}{\dot{y}}$ of this Curve being $=\frac{-ab^2-y^3}{y\sqrt{b^2-y^2}}$ (*Vid. Art.* 48 *and* 57.) we have $\dot{x}=\frac{-ab^2\dot{y}-y^3\dot{y}}{y^2\sqrt{b^2-y^2}}$, and

* Art. 145. therefore $\dot{s}$ ($py^2\dot{x}$*) $=\frac{-pab^2\dot{y}-py^3\dot{y}}{\sqrt{b^2-y^2}}=-\frac{pab^2\dot{y}}{\sqrt{b^2-y^2}}-\frac{py^3\dot{y}}{\sqrt{b^2-y^2}}$. But, in order for the more easy finding the Fluent thereof, put $\sqrt{b^2-y^2}=u$; and then, y being $=\sqrt{b^2-u^2}$, and $\dot{y}=\frac{-u\dot{u}}{\sqrt{b^2-u^2}}$, we shall, by Substitution, get $\dot{s}=\frac{pab^2\dot{u}}{\sqrt{b^2-u^2}}+p\times\overline{b^2\dot{u}-u^2\dot{u}}$.

Whence, the Fluent of $\frac{\dot{u}}{\sqrt{b^2-u^2}}$ being expressed by the Arch (A) of the Circle whose Radius is Unity and

* Art. 142. Sine $\frac{u}{b}$ *, the Fluent of the whole Expression will be $pab^2\times A+p\times\overline{b^2u-\frac{1}{3}u^3}$. Which, when $y=0$, or $u=b$, gives ($pab^2\times\frac{1}{2}p+p\times\frac{2}{3}b^3$) $pb^2\times\overline{\frac{1}{2}pa+\frac{2}{3}b}$ for the Content of the whole Solid, when its Axis becomes infinite.

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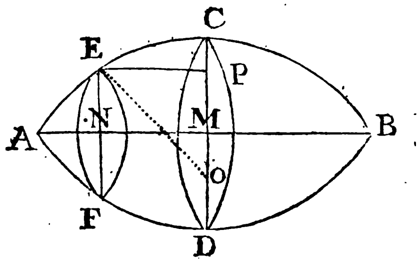
EXAMPLE VII.

152. *Where it is required to find the Content of a parabolic Spindle; generated by the Rotation of a given Parabola* ACB *about its Ordinate* AB.

Put CM (the Abſciſſa of the given Parabola) $= a$, and the Semi-ordinate AM (or BM) $= b$; and, ſuppoſing ENF to be any Section of the Solid parallel to DC, let its Diſtance MN (or EP) from DC, be denoted by w: Then, by the Property of the Curve, we ſhall



have $AM^2\ (b^2) : EP^2\ (w^2) :: CM\ (a) : CP = \frac{aw^2}{b^2}$: Therefore EN $(= CM - CP) = a - \frac{aw^2}{b^2} = \frac{a \times \overline{b^2 - w^2}}{b^2}$, and conſequently $p \times EN^2 = \frac{pa^2}{b^4} \times \overline{b^4 - 2b^2w^2 + w^4} =$ the Area of the Section EF: Which multiply'd by $(\dot{w})$ the Fluxion of MN, gives $\frac{pa^2}{b^4} \times \overline{b^4\dot{w} - 2b^2w^2\dot{w} + w^4\dot{w}}$ for the Fluxion of the Solidity, * whoſe Fluent, $\frac{pa^2}{b^4} \times \overline{b^4w - \frac{2}{3}b^2w^3 + \frac{1}{5}w^5}$, when w becomes $= b$, is $\left(\frac{8pa^2b}{15}\right)$ half the Content of the Solid. * Art. 145.

EXAMPLE VIII.

153. *Let the Solid* ACBD (*see the last Figure*) *be a Spindle, generated by the Rotation of the Segment of a Circle,* ACB, *about its Chord, or Ordinate,* AB.

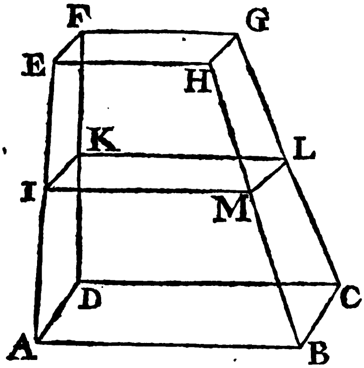
Then, if the Radius OE be put $=r$, OM $=d$, and EP $=w$ &c. (as before) we shall have OP ($=\sqrt{OE^2-EP^2}$) $=\sqrt{r^2-w^2}$, and EN (=OP—OM) $=\sqrt{r^2-w^2}-d$: Therefore $\dot{s}$, in this Case, is $= p\dot{w}\times\overline{\sqrt{r^2-w^2}-d}^{\,2} = p\dot{w}\times\overline{r^2-w^2+d^2-2d\sqrt{r^2-w^2}}$ $= p\dot{w}\times\overline{r^2-d^2-w^2} - p\dot{w}\times\overline{2d\sqrt{r^2-w^2}-2d^2}$: Whence, the Fluent of the *Part*, $p\dot{w}\times\overline{2d\sqrt{r^2-w^2}-2d^2}$ ($=2dp\times\dot{w}\times\overline{\sqrt{r^2-w^2}-d} = 2dp\times\dot{w}\times$ EN) being expressed by $2dp\times$ *Area MNEC* * the Fluent of the *Whole*, or the true Value of s, will be expressed by $pw\times\overline{r^2-d^2-\frac{1}{3}w^2} - 2dp\times$ *Area MNEC*, or by its Equal $p\times MN\times\overline{AM^2-\frac{1}{3}MN^2} - 2p\times OM$ $\times$ *Area MNEC*: Which, when MN $=$ MA, gives $p\times\frac{2}{3}AM^3 - 2p\times OM\times$ *Area ACM*, for the Content of half the Solid: Where the *Area ACM* may be found by *Art.* 124. or more easily by the common Table of the Areas of the Segments of a Circle; to be met with in most Books of *Gauging*.

* Art. 112.

EXAMPLE IX.

154. *Let it be proposed to find the Content of the Solid* AEGB; whose four Sides AH, AF, CH, CF are plane Surfaces, and its Ends ADCB, EFGH given Rectangles, parallel to each other.

Let the Sides AB and AD, of the Base, be denoted by a and b; and those of the Top (EH and EF) by c and d respectively; moreover, let h express the perpendicular Height of the Solid; and let x (consider'd as variable) be the Distance of (IL) any Section thereof (parallel to the Base) from the Plane EG.



It is evident, from the Nature of the Figure, that the Section IL is a Rectangle; and that $b : x :: AB - EH : IM - EH :: BC - HG : ML - HG$.

From these Proportions we have $IM - EH = \frac{\overline{a-c} \times x}{b}$ and $ML - HG = \frac{\overline{b-d} \times x}{b}$: Hence $IM = \frac{\overline{a-c} \times x}{b} + c$, and $ML = \frac{\overline{b-d} \times x}{b} + d$; and consequently the Area of the Rectangle (IL) $= \frac{\overline{a-c} \times \overline{b-d}}{b^2} \times x^2 + \frac{ad - 2cd + cb}{b} \times x + cd$: Which being multiply'd by $\dot{x}$, and the Fluent taken, there results $\frac{\overline{a-c} \times \overline{b-d} \times x^3}{3b^2} + \frac{\overline{ad - 2cd + cb} \times x^2}{2b} + cdx$ for the Content of IFGL:

Which,

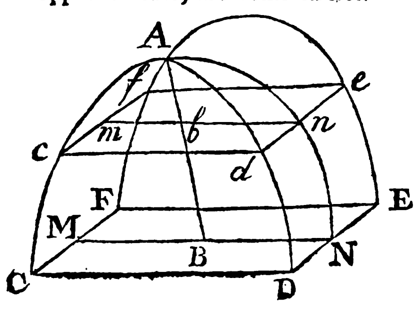
Which, when $x=b$, becomes $\left(\frac{\overline{a-c}\times\overline{b-d}\times b}{3}+\right.$ $\frac{\overline{ad-2cd+cb}\times b}{2}+cdb=\overline{2ab+ad+bc+2cd}\times\frac{1}{6}b=\Big)$ $\overline{AB\times AD+EH\times EF+\overline{AB+EH}\times\overline{AD+EF}}\times\frac{1}{6}b=$ the Quantity propofed to be found.

If EF (d) be fuppofed to vanifh, and the Lines EH and FG to coincide, the Planes AEHB and DFGC will form an Angle or Ridge, at the Top of the Solid (refembling the Roofs of fome Buildings, whofe Ends as well as Sides run up floping) and, in this Cafe, the Content, found above, will become more fimple, being then expreffed by $\overline{2ab+bc}\times\frac{1}{6}b$, or its Equal $\overline{2AB+EH}\times AD\times\frac{1}{6}b$.

But, if EF be fuppofed $=$EH, and AD$=$AB, the Solid will then be the Fruftrum of a fquare Pyramid; and its Content $=\overline{a^2+ac+c^2}\times\frac{1}{3}b$, $=\overline{AB^2+AB\times EH+EH^2}\times\frac{1}{3}b$: From whence, by taking EH$=$o, the Content of the whole Pyramid whofe Bafe is AB^2, and its Altitude b, will alfo be given, being $=AB^2\times\frac{1}{3}b$.

EXAMPLE X.

155\. *Let the propofed Solid be that, commonly known by the Name of a Groin*; whofe Sections parallel to the Bafe are, all, Squares, and whereof the two Sections perpendicular to the Bafe, through the Middle of the oppofite Sides, are Semi-circles.



Let *bcdef* be any Section parallel to the Bafe; and let its Diftance Ab from the Vertex of the Solid, be denoted by x; alfo let a reprefent the Radius AB (or BN) of the cir-

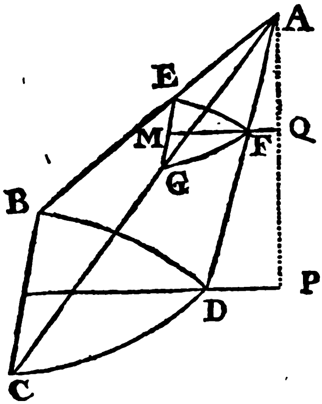
circular Section ABNA, perpendicular to the Base. Then, bn being (by the Property of the Circle) $= \sqrt{2ax - xx}$, the Side of the Square df, will be $= 2\sqrt{2ax - xx}$, and therefore the Area $= 4 \times \overline{2ax - xx}$; whence $\dot{s} = 4\dot{x} \times \overline{2ax - xx}$, and consequently $s = 4ax^2 - \frac{4x^3}{3}$: Which, when $x = a$, becomes $\frac{\overline{2a}|^3}{3}$ = the Content of the whole Solid.

If the Solid be a Groin of any other Kind, or such, that its two Sections perpendicular to the Base, through the Middle of the opposite Sides, are any other Curves than Semi-circles, the Content may, still, be found in the same Manner; and will be always in proportion to the Solid generated by the Revolution of the said Curve about its Axis, as a Square, is to its inscribed Circle. But, if the foresaid perpendicular Sections be Curves of different Kinds, the Sections parallel to the Base will no longer be Squares, but Rectangles; whose Sides are the corresponding (double) Ordinates of the respective Curves. Thus, for Instance, let one Section be a Circle and the other a Parabola, whose Ordinates, to the common Abscissa x, are expressed by $\sqrt{dx - xx}$ and $\sqrt{ax}$, respectively; then the Sides of the rectangular Section, parallel to the Base of the Groin, will be $2\sqrt{dx - xx}$ and $2\sqrt{ax}$: Whence the Area of that Section is $= 4x\sqrt{ad - ax}$, and therefore $\dot{s} = 4x\dot{x}\sqrt{ad - ax}$: Where, by taking the Fluent, * $s =$ * Ar

$$\frac{16d^2\sqrt{ad} - a^{\frac{1}{2}} \times \overline{d - x}|^{\frac{3}{2}} \times \overline{16d + 24x}}{15}$$ = the true Content of such a Solid.

EXAMPLE XI.

156. *Where the Solid* BACD *propoſed is a kind of Cone, or Pyramid*; form'd by conceiving Right-lines to be drawn from every Point in the Perimeter of any given Plane BDC, to a given Point, or Vertex, A above that Plane.



Let EFG be any Section parallel to BDC, whoſe perpendicular Diſtance (AQ) from the Vertex let be denoted by x; moreover, let the whole given Altitude (AP) of the Solid be put $=a$, and the Area of the Baſe BDC (which is alſo ſuppoſed given) $=b$.

In the firſt place, it is eaſy to conceive that the Planes BDC and EFG muſt be ſimilar: And therefore, ſince ſimilar Figures are to each other as the Squares of their like Sides, or Dimenſions, it follows that AP^2 (a^2) : AQ^2 (x^2) :: BDC (b) : EFG $= \frac{bx^2}{a^2}$.

Whence $\dot{s} = \frac{bx^2\dot{x}}{a^2}$, and conſequently $s = \frac{bx^3}{3a^2} = \frac{ba}{3}$, when $x = a$. Therefore the Solidity of a Cone or Pyramid, let the Figure of its Baſe be what it will, is always had by multiplying the Area of the Baſe by $\frac{1}{3}$ of the Altitude.

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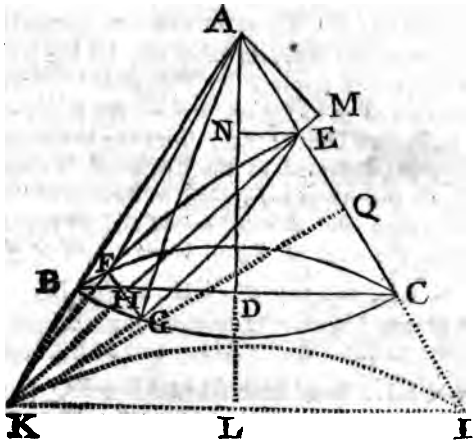
* EXAMPLE XII.

157. *Where it is proposed to find the Content of the Ungula* EFGC, cut off from a given Cone, ABC, by a Plane EFG passing through the Base thereof.



Let AD be the perpendicular Height of the Cone, also let AM be perpendicular to HE, the Axis of the Section FEG, and let FAG be another Section of the Cone, thro' FG and the Vertex A.

Since the Solids CAFG and EAFG, whose Bases are FCG, and FEG, come under the Form specified in the preceding Example, their Contents will therefore be expressed by FCG × $\frac{1}{3}$ AD and FEG × $\frac{1}{3}$ AM respectively: Whose Difference, $\frac{FCG \times AD - FEG \times AM}{3}$, is the Solidity of the *Ungula* CEFG: Where the Bases FCG and FEG being conic Sections, their Areas will be given by *Art.* 115. 124 and 129. from whence the whole will be known. Thus, if HE be supposed parallel to AB, the Section FEG, then being a Parabola, its Area will be = $\frac{2}{3}$ × FG×EH *: Whence the Solidity of the Segment

* Art. 115.

Segment EFGA is $= \frac{2}{3} \times FG \times EH \times AM$: Which being deducted from that of CFGA (found by Help of the common Table of circular Segments) the Remainder will be the Content of the *Ungula*. But, if the Axis EH produced, cuts AB, the Section FEG will be a Segment of an Ellipsis EFKG; whose conjugate Axis (supposing EN and KL perpendicular to AD) is $= 2\sqrt{EN \times KL}$ *. Now, in order to compute the Content, the easiest way, in this Case, let the Ratio of EH to EK (which is given by Trigonometry) be expressed by that of m to Unity, and let the Ratio of CH to CB, be as n to Unity: And from the common Table of *Segments* (adapted to the Circle whose Diameter is Unity) let the Areas answering to the versed Sines m and n, be taken and denoted by M and N respectively: Then, the Area of FEG being $= M \times EK \times 2\sqrt{EN \times KL}$, and that of FCG $= N \times BC^2$ *, the Content of the *Ungula*, by substituting these Values, will become $= \frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times EK \times AM \times 2\sqrt{EN \times KL}$: But, since AM : AE :: KQ (perpendicular to AC) : KE; and AN : AE :: KQ : KI, it follows, by Equality, that $AM \times KE = AN \times KI$; whence the Content of the *Ungula* is also expressed by $\frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times AN \times KI \times 2\sqrt{EN \times KL}$. Which, if H be supposed to coincide with B, and KI with BC, will become $\frac{(0.78539}{3}$ &c. $\times BC^2 \times AD - \frac{0.78539}{3}$ &c. $\times AN \times BC \times 2\sqrt{EN \times BD}) = 0.26179$ &c. $\times BC \times \overline{BC \times AD - 2AN \times \sqrt{EN \times BD}}$.

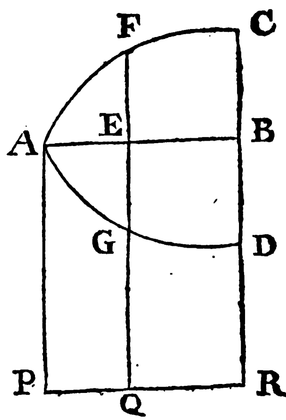
* Art. 41.

* Art. 124 and 130.

When the Section EFG is an Hyperbola, its Area may be found by means of a Table of Logarithms (instead of a Table of Segments) whence the Content of the *Ungula* will likewise be had in that Case.

EXAMPLE XIII.

158. *Let* AFC, *or* AGD, *be a Curve of any Kind*; whose Area, and the Content of the Solid arising from its Rotation about its Axis, or Ordinate, AB, are both known; 'tis proposed to find, from thence, the Content of the Solid generated by the Revolution of that Curve about any other Line PR parallel to the said Axis or Ordinate AB.



Let AP, FQ, and CR be all perpendicular to AB and to the Axis of Motion PQR; also let AP (or EQ) $=a$, AE, consider'd as variable, $=w$, the Area AFE, or AEG $=M$, and the Solid, arising from its Revolution about AB, $=N$. It is plain that the Area of the Circle generated by QF will be $=p\times FQ^2$ * $=p\times\overline{a+EF}^2$ $=pa^2+2pa\times EF+p\times EF^2$; from which deducting the Area, pa^2, generated by QE, the Remainder, $2pa\times EF+p\times EF^2$, will be the Area of the Annulus generated by EF: Whence the Fluxion of the Solid generated by AEF is truly represented by $2pa\times EF\times\dot{w}+p\dot{w}\times EF^2$ ‡: And, in the same manner, it will appear that the Fluxion of the Solid generated by AEG is $2pa\times EG\times\dot{w}-p\dot{w}\times EG^2$. But the Fluent of $EF\times\dot{w}$ (or $EG\times\dot{w}$) is $=$ the Area (M) of AEF (or AEG) *, and that of $p\dot{w}\times EF^2$ (or $p\dot{w}\times EG^2$) equal to (N) the given Solid arising from that Area †; therefore the Fluent of the *Whole*, or the Solidity required, is $2paM+N$, in the former Case, and $2paM-N$ in the latter; where $2pa$, in

* Art. 145.

‡ Art. 145.

* Art. 112.

† Art. 145.

in either Case, expresses the Periphery of the Cylinder described by AB, about the Axis of Rotation PR.

Hence, if ABC and ABD are equal and similar to each other, then the Value of M &c. being the same in both Cases, it follows that the Content of the Solid generated by AFG will be expressed by $2pa \times 2M$, or $2pa \times$ Area AFG.

Now, if (for Example sake) ACD be supposed a Circle, whose Semi-diameter is d, the Area of that Circle being $= pd^2$, the Solid generated by its Revolution (representing the Ring of an Anchor) will therefore be $= 2pa \times pd^2 = 2p^2ad^2$. But if you would know the Content of the Part generated by the upper Semicircle BAC, or the lower one BAD, let the Content

* Art. 148. $\left(\frac{4pd^3}{3}\right)$* of a Sphere whose Semi-diameter is d, be wrote for N, in each of the two foregoing Expressions, and you will then get $p^2ad^2 + \frac{4pd^3}{3}$, and $p^2ad^2 - \frac{4pd^3}{3}$.

Again, if AFC, and AGD be taken as Right-lines, you will have $M = \frac{AB \times BC}{2}$ (or $\frac{AB \times BD}{2}$) and N

* Art. 146. $= p \times BC^2 \times \frac{1}{3} AB$ (or $p \times BD^2 \times \frac{1}{3} AB$)*: Hence the Solid generated by the Triangle ABC is ($= 2pa \times \frac{AB \times BC}{2} + \frac{p}{3} \times BC^2 \times AB$) $= p \times AB \times BC \times \overline{RB + \frac{1}{3} BC}$; and that generated by ABD ($= 2pa \times \frac{AB \times BD}{2} - \frac{p}{3} \times BD^2 \times AB$) $= p \times AB \times BD \times \overline{RB - \frac{1}{3} BD}$.

Lastly, let ABC (or ABD) be considered as a Parabola, whose Ordinate is AB, and Axis CB (or DB):

* Art. 115. Then M being here $= \frac{2}{3} AB \times BC$ (or $\frac{2}{3} AB \times BD$)*

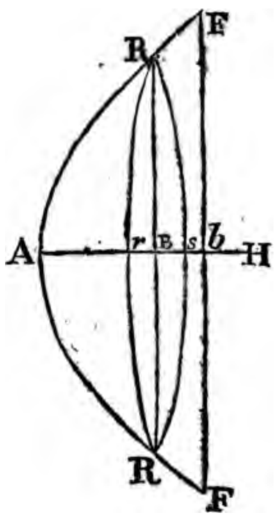
† Art. 152. and $N = \frac{8p}{15} \times AB \times BC^2$ † (or $\frac{8p}{15} \times AB \times BD^2$)

it

it follows that the Solid generated by ABC will be $(= 2pa \times \frac{2}{3} AB \times BC + \frac{8p}{15} \times AB \times BC^2) = 4p \times AB \times BC \times \frac{5BR + 2BC}{15}$, and that generated by ABD $= 4p \times AB \times BD \times \frac{5BR - 2BD}{15}$.

SECTION X.

The Use of Fluxions in finding the Superficies of solid Bodies.



159. LET FAF represent a Solid generated by the Revolution of any given Curve AF about its Axis AH; also let a Circle, whose Diameter is the variable Line (or Ordinate) RBR, be conceived to move uniformly from A towards FF, and to dilate itself so, on all Sides, at the same time, as to generate, by its Periphery, the proposed Superficies RAR: Then, the Length of that Periphery, or the generating Line, being expressed by 3,141592 * &c. $\times$ RR * Art. 142.
$(= 2py)$ and the Celerity with which it moves by $\dot{z}$ * * Art. 135.
the Fluxion of the Superficies RAR, or the Space that *would*

would be uniformly generated in the time of describing $\dot{x}$, will therefore be truly represented by $2py\dot{z}$.

Hence, if w be taken to represent the whole Surface RAR, generated from the Beginning (according to the Method observed in the three last Sections) we shall

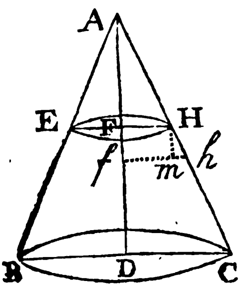
* Art. 135. have $\dot{w} = 2py\dot{z} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$ * ; whence w itself may be found.

EXAMPLE I.

160. *Let it be proposed to determine the convex Superficies of a Cone* ABC.

Then, the Semi-diameter of the Base (BD, or CD) being put $= b$, the slanting Line, or Hypothenuse, AC $= c$, and FH (parallel to DC) $= y$ &c. we shall, from the Similarity of the Triangles ADC and H*mh*,

* Art. 159. have $b : c :: \dot{y}\ (mh) : \dot{z}\ (\mathrm{H}h) = \frac{c\dot{y}}{b}$: Whence $\dot{w}$ ($2py\dot{z}$*) $= \frac{2pcy\dot{y}}{b}$; and consequently $w = \frac{pcy^2}{b}$. This, when



$y = b$, becomes $= pcb = p \times \mathrm{DC} \times \mathrm{AC} =$ the convex Superficies of the whole Cone ABC: Which therefore is equal to a Rectangle under half the Circumference of the Base and the slanting Line.

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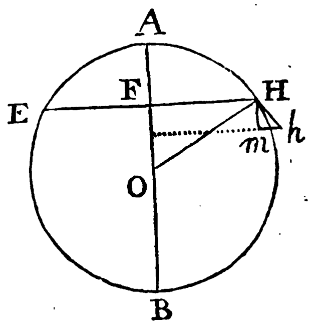
EXAMPLE II.

161. *Let the Solid, whose Surface you would find, be a Sphere* AEBH.

In which Case, putting the Radius OH$=a$, AF$=x$, H$m = \dot{x}$, *&c.* we shall (by reason of the similar Triangles OHF and Hmh *) have y (FH) : a (OH) :: $\dot{x}$ (Hm) : $\dot{z}$ (Hh) $= \frac{a\dot{x}}{y}$: Therefore $\dot{w}$ ($2py\dot{z}$) $=$ * Art. 68.

$2pa\dot{x}$; and consequently the Superficies (w) itself $= 2pax =$ AF $\times$ *Periph.* AEBH. Which, if the whole Sphere be taken, will become AB $\times$ *Periph.* AEBH $=$ four times the Area BEAHO.



Hence the Superficies of a Sphere is equal to four times the Area of its greatest Circle: And the convex Superficies of any Segment thereof, is to that of the *Whole*, as the Axis (or Thickness) of the Segment to the Diameter of the Sphere.

EXAMPLE III.

162. *Wherein let the parabolic Conoid be proposed.*

The Equation of the generating Parabola being $ax = y^2$, or $x = \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2}$ *$) = \sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{a^2}} = \frac{\dot{y}\sqrt{a^2 + 4y^2}}{a}$: * Art. 135.

Hence $\dot{w}$ $(2py\dot{z}) = \frac{2py\dot{y}}{a} \times \overline{a^2 + 4y^2}|^{\frac{1}{2}}$; whereof the

Fluent

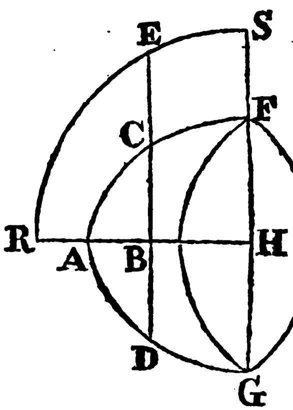
Fluent is $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a}$; which corrected (by sup-

* Art. 79. posing $y=0$ *) gives $\frac{p\times\overline{a^2+4yy}|^{\frac{3}{2}}}{6a}-\frac{pa^2}{6}$, for the Superficies sought.

EXAMPLE IV.

163. *Let it be required to determine the Superficies of a Spheroid.*

Let ACFHG represent one half of the proposed Spheroid, generated by the Rotation of the Semi-ellipsis FAG, about its Axis AH; put AH$=a$, FH (or HG) $=c$, BH$=x$, BC$=y$, FC$=z$, and the Superficies generated by FC (or GD) $=w$: Then, from the Na-



ture of the Ellipsis, we have $y=\frac{c}{a}\sqrt{a^2-x^2}$; whence

* Art. 135. $\dot{y}=-\frac{cx\dot{x}}{a\sqrt{a^2-x^2}}$, and consequently $\dot{z}$ ($=\sqrt{\dot{x}^2+\dot{y}^2}$ *)

$=$

$= \sqrt{\dot{x}^2 + \frac{c^2x^2\dot{x}^2}{a^2 \times \overline{a^2 - x^2}}} = \frac{\dot{x}\sqrt{a^4 - \overline{aa - cc} \times xx}}{a\sqrt{aa - xx}} = \frac{\dot{x}\sqrt{a^4 - b^2x^2}}{a\sqrt{a^2 - x^2}}$ = (by putting (the Excentricity) $\sqrt{a^2 - c^2} = b$) $= \frac{b\dot{x}\sqrt{\frac{a^4}{bb} - x^2}}{a\sqrt{a^2 - x^2}}$: Therefore, in this Caſe, $\dot{w}$ $(2py\dot{z}) = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{bb} - x^2}$; whoſe Fluent, in an Infinite Series, is $2pcx \times \overline{1 - \frac{b^2x^2}{2.3a^4} - \frac{b^4x^4}{2.4.5a^8} - \frac{3b^6x^6}{2.4.6.7a^{12}}}$. But the ſame Fluent may be, *otherwiſe*, very eaſily exhibited by means of the Area of a Circle: For, if from the Center H, with a Radius equal to $\frac{aa}{b}$, a Circle SER be deſcribed, and the Ordinate BC be produced to interſect it in E, it is evident that BE $= \sqrt{\frac{a^4}{bb} - xx}$, and that the Fluxion of the Area ESHB will be expreſſed by $\dot{x}\sqrt{\frac{a^4}{bb} - x^2}$; which being to $\frac{2pbc\dot{x}}{aa} \times \sqrt{\frac{a^4}{bb} - x^2}$, the Fluxion before found, in the conſtant Ratio of 1 to $\frac{2pbc}{a^2}$, their Fluents muſt therefore be in the ſame Ratio; and ſo the latter, expreſſing the Superficies CFGD, will conſequently be $= \frac{2pbc}{aa} \times$ BESFH $= 2p \times \frac{\text{FH}}{\text{HS}}$ $\times$ BESFH.

This Solution, it may be obſerved, obtains only in Caſe of an *oblong* Spheroid, generated by the Rotation of the Ellipſis about its greater Axis; for, in an *oblate* Spheroid,

Spheroid, generated about the leffer Axis, the Value of b $(\sqrt{a^2-c^2})$ will be impoffible; fince, in this Cafe HF is greater than HA. But, if we, *here*, put $b = \sqrt{c^2-a^2}$, and $d = \frac{a^2}{b}$; the Value of $\dot{w}$ (found above) will become $= \frac{2pbc\dot{x}}{a^2}\sqrt{\frac{a^4}{bb}+x^2} = \frac{2pc\dot{x}}{d}\sqrt{d^2+x^2}$ $= \frac{2pc}{d} \times \dot{x}\sqrt{d^2+x^2}$: Whofe Fluent may be brought out by help of a Table of Logarithms: For, let the variable Part $\dot{x}\sqrt{d^2+x^2}$ be transformed to $\left(\frac{\dot{x}\times\overline{d^2+x^2}}{\sqrt{d^2+x^2}} = \frac{d^2\dot{x}+x^2\dot{x}}{\sqrt{d^2+x^2}} = \frac{d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}}\right.$ $\left.=\right)$ $\frac{\frac{1}{2}d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}} + \frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2+x^4}}$, fo that the Numerator of the firft Term $\frac{\frac{1}{2}d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}}$ (now in a given Ratio to the Fluxion of the Quantity under the radical Sign) may be had by the common Rule*; by which means we get $\frac{1}{2}\sqrt{d^2x^2+x^4}$, for the true Fluent of the faid Term; to which adding the Fluent of the other Term $\frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2+x^4}}$, or $\frac{\frac{1}{2}d^2\dot{x}}{\sqrt{d^2+x^2}}$ (given by *Art.* 126.) there arifes $\frac{1}{2}x\sqrt{d^2+x^2}+\frac{1}{2}d^2 \times$ hyp. Log. $x+\sqrt{d^2+x^2}$, for the Fluent of $\dot{x}\sqrt{d^2+x^2}$: And this, corrected * and multiply'd by $\frac{2pc}{d}$, gives $\frac{pcx}{d}$ $\sqrt{d^2+x^2} + pcd \times$ hyp. Log. $\frac{x+\sqrt{dd+xx}}{d}$, for the Superficies in this Cafe, where the propofed Spheroid is an oblate One.

* Art. 77.

* Art. 78.

EX.

EXAMPLE V.

164. *Let the Solid, whose Superficies is sought, be the hyperbolical Conoid.*

Let the semi-transverse Axis, of the generating Hyperbola, $=a$, the semi-conjugate $=c$, and the Distance of any Ordinate from the Center thereof $=x$; then from the Nature of the Curve you will have $y = \frac{c}{a}\sqrt{x^2-a^2}$; whence $\dot{y} = \frac{c x\dot{x}}{a\sqrt{xx-aa}}$, $\dot{z} = \frac{\dot{x}\sqrt{a^2+c^2 \times x^2 - a^4}}{a\sqrt{xx-aa}}$, and $\dot{w}$ $(2p y\dot{z}) = \frac{2pc\dot{x}}{aa} \times \sqrt{aa+cc \times xx - a^4}$; which last Value, if d^2 be put $= \frac{a^4}{a^2+c^2}$, will be more commodiously expressed by $\frac{2pc\dot{x}}{d}\sqrt{x^2-d^2}$: Whereof the Fluent, by proceeding as in the latter Part of the foregoing Example, will come out $= \frac{pcx\sqrt{xx-dd}}{d} - pcd \times$ *hyp. Log.* $x+\sqrt{x^2-d^2}$: Which corrected (by taking $x=a$) becomes $\frac{pcx}{d}\sqrt{xx-dd} - pc^2$, $-pcd \times$ *hyp. Log.* $\frac{x+\sqrt{x^2-d^2}}{a+\frac{cd}{a}}$, the true Measure of the required Superficies.

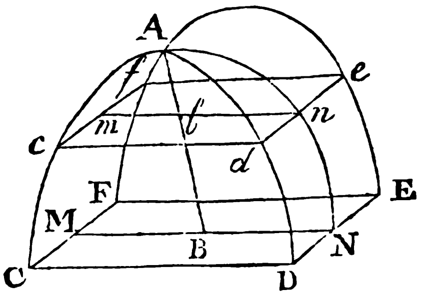
EXAMPLE VI.

165. *Let it be proposed to find the Superficies of the Solid called a Groin.* (Vid. Art. 155.)

Let $bcdef$ be any Section of the Solid parallel to the Base thereof, and let x denote its Distance from the

O Vertex

Vertex A, alſo put z equal to the correſponding Arch An of the ſemi-circular Section NnA &c. whoſe Radius AB or BN let be denoted by a.



It appears from *Art.* 161. that $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax-xx}}$: Which Value, multiply'd by $(2\sqrt{2ax-xx})$ that of de $(=2bn)$ gives $2a\dot{x}$ * for the Fluxion of one of the four equal convex Superficies by which the Solid is bounded. Hence the whole Superficies (excluding the Baſe) comes out $= 8a^2$:. Which therefore is exactly equal to twice the Baſe.

* Art. 159.

If the Solid be ſuppoſed a Groin of any other Kind, ſuch that its two equal Sections, thro' the Middle of the oppoſite Sides, are other Curves than Circles, the Superficies may *ſtill* be had in the ſame manner; and will be always in proportion to the Superficies ariſing from the Revolution of either of the ſaid equal Curves about its Axis, as a Square is to its inſcribed Circle. Thus, the Superficies of a parabolic Conoid being $=$

$$\frac{p\times\overline{aa+4yy}^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$$ (*by Art.* 162.) the convex Superficies of the Groin, ſuppoſing the generating Curve AnN to be a Parabola, will therefore be $=$

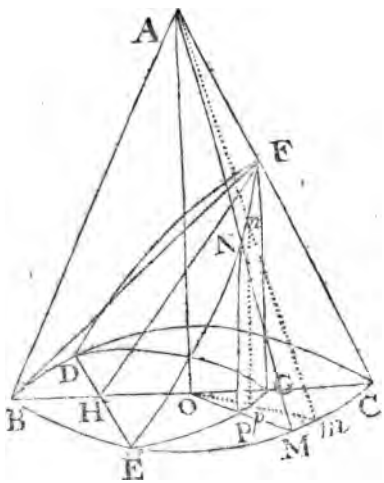
$$\frac{4\times\overline{a^2+4yy}^{\frac{3}{2}}}{6a} - \frac{4a^2}{6}.$$

EX-

EXAMPLE VII.

166. *Wherein let it be required to find the convex Superficies of a conical* Ungula ECFD; formed by a Plane DFE passing thro' the Base of the Cone.

Let a right-angled Triangle AOM (whose Base OM is the Radius of the Circle BDCE) be supposed to revolve about the Axis AO; whilst a Right-line NP, drawn perpendicular to OM from the Intersection of AM and the Arch EFD, traces out, upon the Base of the Cone, the curve-line EPGD.



If MPOAN and *mp*OA*n* be considered as two Positions of the generating Triangle indefinitely near to each other, it is evident that the Space MA*m*, generated by AM, will be to the Space MO*m*, generated by OM, as AM to OM, or OB. Whence, MN and MP being proportional Parts of AM and OM (because NP is parallel to AO) it is likewise plain that the Spaces MN*nm* and MP*pm*, generated by those Parts, will be to each other in the same Ratio of AM to OB. And, since this every where holds, it follows that the whole Space (ENM) &c. generated by MN, will be to that (EPM) generated by PM, as AM to OB: And so the whole required Superficies (generated by AM) is truly represented by $\frac{AM}{OB}$ × Area EPGDCE.

But now, to find this Area, EPGDCE, it is observable that the Area of the Plane DFE (being the Segment of a Conic-section) is given, *by Art.* 115. 129 or 130. And it is very easy to apprehend and demonstrate that the Area so given will be to that of EGDH, as the Radius to the Co-sine of the Angle of the Inclination of the said Plane to the Base, or as HF to HG. Therefore, seeing EGDH is $= \frac{HG}{HF} \times EFD$, we have EPGDCE ($= ECDHE - EGDH$) $= ECDHE - \frac{HG}{HF} \times EFD$; and consequently $\frac{AM}{OB} \times EPGCDE = \frac{AM}{OB} \times ECDHE - \frac{AM \times HG}{OB \times HF} \times EFD =$ the convex Superficies that was to be found.

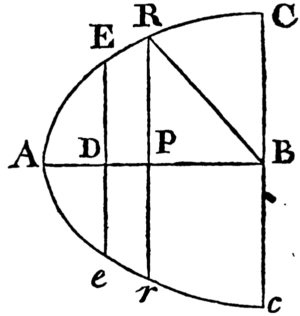
If the Point H be supposed to coincide with B, ECDHE will become the *whole* Circle CB; and EDF will become a whole Ellipsis, whose greater Axis is BF, and its lesser Axis $= 2\sqrt{OB \times OG}$. * Therefore, the Area of the former Figure will be expressed by $p \times BO^2$ †, and that of the latter by $p \times \frac{1}{2} BF \times \sqrt{OB \times OG}$; and so the convex Superficies of the Part BFC will be ($= \frac{AM}{OB} \times p \times BO^2 - \frac{AM \times BG}{OB \times BF} \times p \times \frac{1}{2} BF \times \sqrt{OB \times OG}$) $= p \times AM \times OB - p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$: Which being deducted from ($p \times AM \times OB$) the Superficies of the whole Cone BAC, there rests $p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$, for the Superficies of the oblique Cone BAF; which from hence is also given.

* Art. 41.
† Art. 124.

SCHO-

SCHOLIUM.

167. In moſt of the Examples, delivered in the four laſt Sections, the Part of the propoſed *Figure* next the Vertex, whether, a Curve, Solid, or Superficies, is firſt found; from whence, by taking the Altitude (x) of that Part equal to (a) the Altitude given, the Content of the *Whole* is deduced: But, if the Content of the lower Segment (BCED) of any Figure (ABC) ariſing by taking away a Part (ADE) next the Vertex, be required; then the Difference between the *Whole* and the Part taken away (found as before explained) will be the Quantity ſought.



Thus, for Example, let ABC be the common *Parabola*, and let it be propoſed to find the Content of the Part, BCED, included between any two Ordinates BC (b) and DE (c) at a given Diſtance BD (d) from each other: Then, the Equation of the Curve being $ax = y^2$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $y\dot{x}$ * $= \frac{2y^2\dot{y}}{a}$, whoſe Fluent $\frac{2y^3}{3a}$ is a general Expreſſion for the Area comprehended between the Vertex and the Ordinate y: Whence, expounding y, by b and c ſucceſſively, we get $\frac{2b^3}{3a}$ and $\frac{2c^3}{3a}$ for the correſponding Values of ABC and ADE; whoſe Difference $\frac{2b^3 - 2c^3}{3a}$ is the required Area BCED: But, to expreſs the ſame independent of a, it will be, by the Property of the Curve, $b^2 : c^2 :: AB : AD$; whence,

* Art. 112.

whence, by Division, $b^2 : b^2 - c^2 :: AB : BD\ (d)$ and consequently $\frac{b^2 - c^2}{d} = \frac{b^2}{AB} = a$; which first Value being wrote instead of a, there results $BCED = \frac{\overline{2b^3 - 2c^3} \times d}{3b^2 - 3c^2} = \frac{2d}{3} \times \frac{b^2 + bc + c^2}{b+c}$.

After the same Manner, the Segments of other Figures may be found; but in many Cases they will be more readily had from a direct Investigation, without either finding the Whole or the Part taken away.

Thus, in the Case above, if the Excess of any Ordinate RP above DE (c) be denoted by w, we shall have, by the Property of the Curve, $b^2 - c^2\ (BC^2 - DE^2) : \overline{c+w}^2 - c^2\ (RP^2 - DE^2) :: DB\ (d) : DP = \frac{d \times \overline{2cw + w^2}}{b^2 - c^2}$; whose Fluxion $\left(d \times \frac{2c\dot{w} + 2w\dot{w}}{b^2 - c^2}\right)$ multiply'd by $c + w$ $(= PR)$ gives $d \times \frac{2c^2\dot{w} + 4cw\dot{w} + 2w^2\dot{w}}{b^2 - c^2}$, for the Fluxion of the Area DPRE: Whereof the Fluent (which is $2dw \times \frac{c^2 + cw + \frac{1}{3}w^2}{b^2 - c^2}$) will, when $w = b - c$ (or $RP = BC$) be truly expounded by $\frac{2d \times \overline{b - c} \times \overline{\frac{1}{3}b^2 + \frac{1}{3}bc + \frac{1}{3}c^2}}{b^2 - c^2}$ or its Equal, $\frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$; *the same as before.*

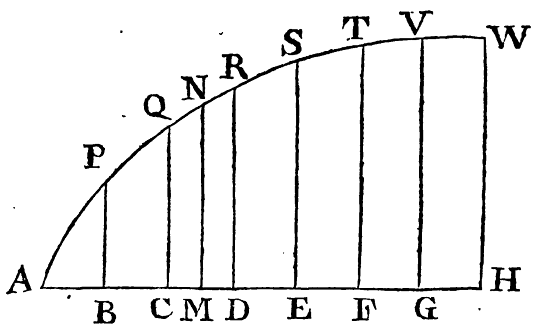
Again, for another Example, let CEDec be consider'd as the lower Frustrum of an Hemisphere, whose Center is the Point B: Then, BP being, *here*, denoted by w, we shall have $y^2\ (= BR^2 - BP^2) = b^2 - w^2$, and consequently $py^2\dot{w}$ * $= p \times \overline{b^2\dot{w} - w^2\dot{w}}$; whose Fluent

* Art. 145.

Fluent $(p \times \overline{b^2 w - \frac{1}{3} w^3} = \frac{1}{3} pw \times \overline{3b^2 - w^2} = \frac{1}{3} pw \times \overline{2b^2 + b^2 - w^2} = \frac{1}{3} pw \times \overline{2b^2 + y^2} = \frac{1}{3} p \times BP \times \overline{2BC^2 + PR^2})$ is the true Content of the Part CED*ec*; which will alſo hold when the Figure is a Spheroid.

This laſt Method, of finding the Content of a Portion of a Figure, remote from the Vertex, will be of Service, when the general Value, for the *Whole*, cannot be expreſſed without an Infinite Series; becauſe ſuch a Series, in that Caſe, not coverging, becomes uſeleſs *.

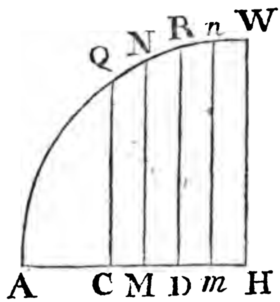
By dividing the whole propoſed Figure, AHW, into a Number of ſuch Portions, HV, GT, FS, &c. the Content thereof may be obtained, when to find it at once, by a Series, commencing from the Vertex, would be altogether impracticable. * Art. 93.



But, to render ſuch an Operation as ſhort and eaſy as may be, it will be proper to find each Part (DQ, &c.) of the Figure, by means of a Series proceeding both Ways, from the middle Ordinate (MN) between the two correſponding Extremes (CR and DR.)

Thus, let the Value of MN (found by the Property of the Curve) be denoted by a; and let the Value of DR, in a Series, be repreſented by $a + bx + cx^2 + dx^3 + ex^4 + fx^5 +$ &c. where $x = MD$; then the Area MDRN will be repreſented by the Fluent of $\dot{x} \times \overline{a + bx + cx^2 + dx^3 +}$ &c.

&c. or by $x\times\overline{a+\frac{bx}{2}+\frac{cx^2}{3}+\frac{dx^3}{4}+}$ &c. And, by writing $-x$ instead of x, the Ordinate CQ will be expressed by $a-bx+cx^2-dx^3$ &c. and the Area MCQN, by $x\times\overline{a-\frac{bx}{2}+\frac{cx^2}{3}-\frac{dx^3}{4}+\frac{ex^4}{5}}$ &c. whence the Area CDRQ is $=2x\times\overline{a+\frac{cx^2}{3}+\frac{ex^4}{5}+\frac{gx^6}{7}+}$ &c. Therefore, if DE, EF, FG, and GH be supposed, each, $=$ BC $(2x)$ and the Areas DS, ET, &c. (found as above) be denoted by $2x\times\overline{a'+\frac{c'x^2}{3}+\frac{e'x^4}{5}}$ &c. and $2x\times\overline{a''+\frac{c''x^2}{3}+\frac{e''x^4}{5}}$ &c. respectively, it follows that the Area CR + DS + ET will be represented by $2x\times\overline{a+a'+a''}$ &c. $+\frac{2}{3}x^3\times\overline{c+c'+c''}$ &c. $+\frac{2}{5}x^5\times\overline{e+e'+e''}$ &c.



An Example will shew the Use of this last Expression: Let CHWQ be a Portion of a Quadrant HAW of a Circle, whose Base HC (conceived to be divided into four equal Parts) is equal half the Radius AH, represented by *Unity*. Then, putting CM $(=\text{DM}=\text{D}m=m\text{H}=\frac{1}{8})$ $=x$, HM $(=\frac{3}{8})=p$, and Hm $(=\frac{1}{8})=q$, we have, by the Property of the Circle, a (MN) $=\sqrt{\text{HN}^2-\text{HM}^2}=\sqrt{1-pp}$, and DR

DR $(=\sqrt{HR^2-HD^2})=\sqrt{1-\overline{p-x}|^2}=\sqrt{1-p^2+2px-x^2}=\sqrt{a^2+2px-x^2}$; which, in a Series, is $(=a+\frac{2px-x^2}{2a}-\frac{\overline{2px-x^2}|^2}{8a^3}+\&c.)$ $=a+\frac{px}{a}-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}\times x^2$ &c. Therefore, in this Case, $b=\frac{p}{a}$, $c=-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}$, &c. Which Value of c, by writing $1-a^2$ for its Equal p^2, will be reduced to $-\frac{1}{2a^3}$. From whence it is also evident that $\dot{c}=-\frac{1}{2\dot{a}^3}$ (supposing $\dot{a}$ $(mn)=\sqrt{1-q^2}$)

Consequently $2x\times\overline{a+\acute{a}+\acute{\acute{a}}\ \&c.}+\frac{2}{3}x^3\times\overline{c+\acute{c}+\acute{\acute{c}}\ \&c.}+\frac{2}{5}x^5\times\overline{e+\acute{e}+\acute{\acute{e}}\ \&c}$ $(=\overline{a+\acute{a}}\times 2x+\overline{c+\acute{c}}\times\frac{2x^3}{3})=\overline{\sqrt{\frac{55}{64}}+\sqrt{\frac{63}{64}}}\times\frac{1}{4}-$

$$\overline{\frac{1}{\frac{2\times 55}{64}\sqrt{\frac{55}{64}}}+\frac{1}{\frac{2\times 63}{64}\sqrt{\frac{63}{64}}}}\times\frac{2}{64\times 8\times 3}=$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{1}{3\times 55\sqrt{55}}-\frac{1}{3\times 63\sqrt{63}}=$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{\sqrt{55}}{3\times 55\times 55}-\frac{\sqrt{63}}{3\times 63\times 63}=$$

0,48730 = the Area, CHWQ, that was to be found.

This Example, chosen as an Illustration of the foregoing Method, may indeed be wrought the common Way; whence the very same Conclusion is brought out (*Vide*

(*Vide Art.* 124.) But that Method is also applicable to any other Case, whether the Part proposed be near to the Vertex, or remote from it; and whether the Figure itself be a Curve, Solid or Superficies; since the Measure thereof may, always, be expressed by the Area of a Curve.

There is another Way, well known to Mathematicians, whereby the Area of a Curve may be determined, by means of a Number of equidistant Ordinates; which Method, derived from *that of Differences*, may, also, be used to good Purpose, in Cases like those above specified: But, it having been treated of by several *others*, and also in my *Dissertations*, the Reader will excuse me, if no further Notice is taken of it here.

SECTION XI.

Of the Use of FLUXIONS *in finding the Centers of Gravity, Percussion, and Oscillation of Bodies.*

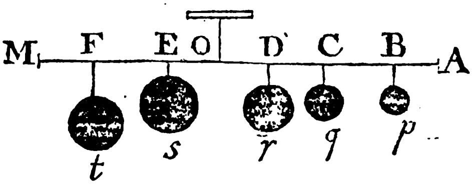
168. THE Center of Gravity is that Point of a Body, by which, if it were suspended, it would rest *in Equilibrio*, in any Position.

LEMMA.

169. *Let p, q, r, s, &c. be any Number of given Weights, hanging at an inflexible Line (or Rod)* AM *suspended* in Equilibrio, *in an horizontal Position, at the Point* O; *to determine the Position of that Point.*

Since (*by Mechanics*) the Force of any Weight (p) to raise the opposite End (M) of the Balance, is as that Weight drawn into its Distance (BO) from the Fulcrum,

crum, we shall, from the Equality of these Forces, have $p \times OB + q \times OC + r \times OD = s \times OE + t \times OF$,



that is $p \times \overline{AO - AB} + q \times \overline{AO - AC} + r \times \overline{AO - AD} = s \times \overline{AE - AO} + t \times \overline{AF - AO}$, and consequently $AO = \frac{p \times AB + q \times AC + r \times AD + s \times AE + t \times AF}{p + q + r + s + t}$.

From which it appears, that, *if each Weight be multiply'd by its Distance from the End (or any given Point) of the Axis, the Sum of all the Products divided by the Sum of all the Weights, will give the Distance of the Center of Gravity from that End (or Point.)*

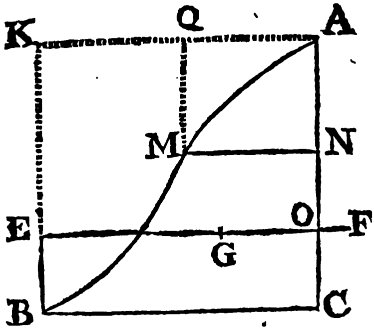
Note. The Products here mentioned are, usually, call'd *the Forces*, of their respective Weights; not in respect to their Action at the Center O (which is expressed by a different Quantity) but with regard to the Effects they have in the Conclusion, or the Value of AO; which appear to be in that Ratio.

PROPOSITION I.

170. *To determine the Center of Gravity of a Line, Plane, Superficies, or Solid* (admitting the three former capable of being affected by Gravity.)

Let AMBC be the proposed Figure, and G the Center of Gravity thereof; thro' which, parallel to the Horizon, let the Line EF be drawn, intersecting AC, at Right-angles, in O; also let AK and NM be perpendicular to AC, and parallel to EF.

171. Case



171. Case 1. *If the Figure* AMBC *be a Plane*; let it be supposed to rest *in Equilibrio* upon the Line EF; and then, if the Line MN be consider'd as a Weight, its Force (*defined above*) will be expressed by MN drawn into its Distane (AN) from the End of the Axis AC; that is by yx (supposing, as usual, AN $= x$ and MN$=y$.) This, therefore, multiply'd by the Fluxion of AN, gives $yx\dot{x}$ for the Fluxion of the Force of the Plane AMN; whose Fluent, when $x =$ AC, expresses the Force of the whole Plane, or the Sum of all the Products of the Ordinates (or Weights) by their respective Distances from AK: Which Fluent being, therefore, divided by the Area ABC, or the Fluent of $y\dot{x}$ (according to the foregoing Lemma) the Quotient $\left(\frac{Flu.\,yx\dot{x}}{Flu.\,y\dot{x}}\right)$ will give (AO) the Distance of the Center of Gravity from the Line AK.

172. Case 2. *If the Figure be a Solid*; let MN be a Section thereof by a Plane perpendicular to the Horizon; then, the Area of that Section being denoted by A, the Force thereof (consider'd as above) will be expressed by Ax, and the Fluxion of the Force of the Solid AMN by $Ax\dot{x}$; whose Fluent, divided by the Content of the Body, or the Fluent of $A\dot{x}$, gives AO, in this Case. But, if the Solid be the half (or the whole) of that arising from the Rotation of a Curve AMB about its Axis AC; then (putting p for the Area of the Circle whose Radius is Unity) A will become $= \frac{1}{2}py^2$ *; and consequently AO $= \frac{Flu.\,\frac{1}{2}py^2x\dot{x}}{Flu.\,\frac{1}{2}py^2\dot{x}} = \frac{Flu.\,y^2x\dot{x}}{Flu.\,y^2\dot{x}}$.

* Art. 145.

173. Case

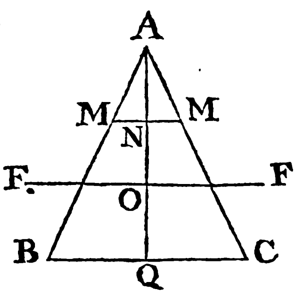
173. Case 3. *If the Figure proposed be the Curve-line* AMB; then, the Force of a Particle at M being expressed by AN or MQ (x) we shall (putting AM $= z$) have $\frac{Flu.\ x\dot{z}}{z} =$ AO.

174. Case 4. *But if the Figure given be the Superficies generated by the Rotation of* AMB *about* AC.

Then, the Periphery of the Circle generated by the Point M being $= 2py$, it follows that $\frac{Flu.\ 2pyx\dot{z}}{Flu.\ 2py\dot{z}} = \frac{Flu.\ yx\dot{z}}{Flu.\ y\dot{z}} =$ AO.

EXAMPLE I.

175. *Let the Figure proposed be the isosceles Triangle* ABC.



It is evident the Center of Gravity (O) will be somewhere in the Perpendicular AQ: And, if AQ $= a$, BC $= b$, AN $= x$, and MM $= y$; then y being $= \frac{bx}{a}$, we shall have, *by Case* 1, AO $\left(= \frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}}\right) = \frac{Flu.\ x^2\dot{x}}{Flu.\ x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2x}{3} = \frac{2}{3}$ AQ, when $x =$ AQ; and consequently OQ $= \frac{AQ}{3}$.

In the very same manner, the Center of Gravity of any other (plane) Triangle will appear to be at $\frac{2}{3}$ of the Altitude of the Triangle.

EX-

EXAMPLE II.

176. *Let the Figure proposed be a Parabola of any Kind;* whereof the Equation is $y = \frac{x^n}{a^{n-1}}$.

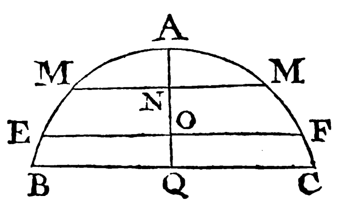
Here, $\frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}} = \frac{Flu.\ x^{n+1}\dot{x}}{Flu.\ x^n\dot{x}} = \frac{n+1}{n+2} \times x =$ the Diſtance of the Center of Gravity from the Vertex of the Curve.

EXAMPLE III.

177. *Let* BAC *be a Segment of a Circle.*

Then, if the Radius thereof be put $= r$, we ſhall have y (NM) $= \sqrt{2rx - xx}$: Whence the Fluent of $yx\dot{x}$ $(x\dot{x}\sqrt{2rx - xx})$ will, by *Art.* 163. be found $= -\frac{\overline{2rx - xx}^{\frac{3}{2}}}{3} + r \times$ Area ANM; which divided by ANM, gives $r - \frac{NM^3}{3 \times \text{Area ANM}} = AO$*. This, therefore, when BAC is a Semi-circle, becomes $= \frac{576}{1000} \times r$, nearly.

* Art. 171.



But, with reſpect to the Center of Gravity of the Arch BAC; we have, $Flu.\ x\dot{z}$, (*by Caſe* 3.) = Fluent of $\frac{rx\dot{x}}{\sqrt{2rx - xx}} = r \times \overline{AM - MN}$; and conſequently AO *here* $= r - \frac{r \times MN}{AM}$.

EX.

EXAMPLE IV.

178. *Let* ABC (see the preceding Figure) *represent a Segment of a Sphere, or Spheroid.*

In which Case, denoting the Axis of the Sphere, or Spheroid, by a, and the other Axis of the generating Curve, when an Ellipsis, by b, we have $y^2 = \frac{bb}{aa} \times \overline{ax - xx}$;

and therefore $\frac{\text{Flu. } y^2 x\dot{x}}{\text{Flu. } y^2 \dot{x}} * = \frac{\text{Flu. } \overline{ax - xx} \times x\dot{x}}{\text{Flu. } \overline{ax - xx} \times \dot{x}} =$ * Art. 172.

$$= \frac{\frac{1}{3} ax^3 - \frac{1}{4} x^4}{\frac{1}{2} ax^2 - \frac{1}{3} x^3} = \frac{\frac{1}{3} ax - \frac{1}{4} x^2}{\frac{1}{2} a - \frac{1}{3} x} = \frac{x \times \overline{4a - 3x}}{6a - 4x} = \text{AO}.$$

If the Solid be an hyperbolical Conoid, the Distance (AO) of its Center of Gravity from the Vertex, will also be exhibited by the Expression here brought out, when the negative Signs are changed to positive ones.

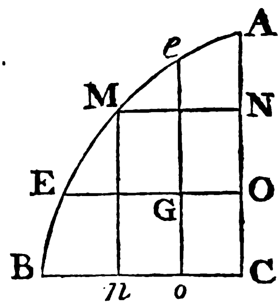
179. In those Cases where the Figure cannot be divided into two Parts, equal and like to each other (as a Curve is by its Axis, *&c.*) the Position of two Lines EO, *eo* (*see the ensuing Figure*) must be determined, *as above*; in whose Intersection (G) the Center of Gravity will be found.

EXAMPLE V.

Let ABC *be a Semi-parabola of any Kind*; whereof the Equation is $y = \frac{x^n}{a^{n-1}}$.

It appears, from *Ex.* 2. that (AO) the Distance of EGO from the Vertex, is expressed by $\frac{n+1}{n+2} \times$ AC: But to find the Position of *o*G*e* (perpendicular to EO) let M*n* be parallel to *eo*, or AC; then, AN being $= x$, and

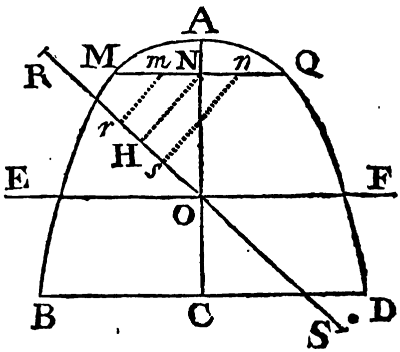
and NM $(y) = \frac{x^n}{a^{n-1}}$, if AC be denoted by b, we



ſhall have $Mn = b - x$, and $Mn \times NM \times \dot{y} = \overline{b - x} \times \frac{x^n}{a^{n-1}} \times \frac{nx^{n-1}\dot{x}}{a^{n-1}} = \frac{nbx^{2n-1}\dot{x} - nx^{2n}\dot{x}}{a^{2n-2}}$, for the Fluxion of the Sum of the Forces in this Caſe (*Vid. Art.* 171.) whoſe Fluent $\left(\frac{nbx^{2n}}{2na^{2n-2}} - \frac{nx^{2n+1}}{\overline{2n+1} \times a^{2n-2}}\right.$ $= \frac{x^{2n}}{a^{2n-2}} \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} = y^2 \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} =$ $y^2 \times \frac{b}{4n+2}$, or $\frac{BC^2 \times AC}{4n+2}$, when $x = b$) divided by the Area ABC $\left(= \frac{BC \times AC}{n+1}\right)$ gives $\frac{n+1}{4n+2} \times$ BC for the true Value of Co, or OG. Which, in caſe of the common Parabola, where $n = \frac{1}{2}$, and where AO $\left(\frac{n+1}{n+2} \times AC\right) = \frac{3}{5}AC$, will become $= \frac{3}{8}CB$.

Before I leave this Subject it may not be improper to take notice, *that*, whatever Line you found your Calculations upon, by ſuppoſing the Figure to reſt, *in Equilibrio*,

Equilibrio, upon that Line, the very ſame Point, for the Place of the Center of Gravity, will be determined.



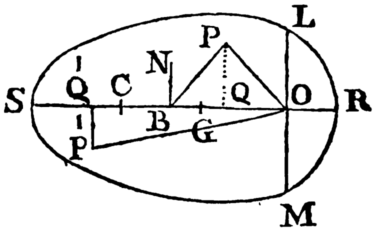
180. Thus, let O be the Point in the Axis AC, of a given Curve BAD, determined, as above, by ſuppoſing the Figure to reſt upon EF perpendicular to AC; and let RS be any other Line paſſing through the Point O; then I ſay the Sum of the *Momenta* of the Particles on each Side of RS will, *alſo*, be equal. For, if from two Points, in any Ordinate MQ, equally diſtant from the middle Point N, two Perpendiculars *mr* and *ns* be let fall upon RS, the Efficacy of thoſe two Points, in reſpect to RS, will be repreſented by $mr+ns$, or its Equal 2NH (ſuppoſing NH alſo perpendicular to RS.) Whence the Efficacy of all the Particles in MQ, will be expreſſed by their Number multiply'd by NH, or by MQ×NH: Which is to their Efficacy (MQ×ON) when referred to the Line EF, in the conſtant Ratio of NH to ON, or of the Sine of the Angle RON to Radius. Whence it is evident that the Force of all the Ordinates (or the whole Curve) in the former Caſe, muſt be to that in the latter, in the ſame Ratio: But the ſaid Force, in the one Caſe, is equal to nothing by Hypotheſis, therefore it muſt be likewiſe ſo in the other: And conſequently the Sum of the *Momenta* of the Particles, on each Side of RS, equal to each other.

Much after the ſame manner the thing may be proved, in a Solid: Whence it will appear that there is actually ſuch a (fix'd) Point in a Body as the Center of Gravity is defined to be: Which, however evident from mechanical Conſiderations, is not ſo eaſy to demonſtrate, *geometrically*, from the Reſolution of Forces.

PROPOSITION II.

181. *To determine the Center of Percussion of a Body.*

The Center of Percussion is that Point, in the Axis of Suspension of a vibrating (or revolving) Body, at which it may be stopt, by an immoveable Obstacle, so as to rest thereon *in Equilibrio* as it were, without acting upon the Center of Suspension.



Let O be the Point of Suspension, G the Center of Gravity, and SLM a Section of the Body, by the Plane wherein the Axis of Suspension OGS performs its Motion; to which Section let all the Particles of the Body be conceived to be transferred in such Parts thereof where they would be projected into (*orthographically*) by Lines parallel to the Axis of Motion; which Supposition will neither affect the Place of the Center of Gravity nor the angular Motion of the Body.

Since the angular Velocity of any Particle P is as the Distance, or Radius, OP, its Force in the Direction, PB, perpendicular to OP, will be expressed by $P \times OP$. Therefore the Efficacy of that Force upon the Axis, at B, in the perpendicular Direction BN (supposing the Axis stopt at C the Center of Percussion) will be $P \times OP \times \frac{OP}{OB}$, whose Power to turn the Body about the Point C is therefore as $P \times OP \times \frac{OP}{OB} \times BC = P \times \frac{OP^2 \times BC}{OB} = P \times \frac{OP^2 \times \overline{OC - OB}}{OB} = P \times \frac{OP^2 \times OC}{OB} - P \times OP^2$; which, if PQ be made perpendicular to OS, OS, will at last (because $\frac{OP^2}{OB} = OQ$) be reduced to $P \times OQ \times OC - P \times OP^2$. By the very same Argument, the Force of any other Particle P' will be denoted by $P' \times OQ' \times OC - P' \times OP'^2$ *&c. &c.* But, as all these Forces must destroy one another (by the Nature of the Problem) the Sum of all the Quantities $P \times OQ \times OC$, $P' \times OQ' \times OC$, *&c.* must therefore be = the Sum of all the Quantities $P \times OP^2$, $P' \times OP'^2$ *&c.* and consequently $OC = \frac{P \times OP^2 + P' \times OP'^2 + \text{\&c. \&c.}}{P \times OQ + P' \times OQ' + \text{\&c. \&c.}}$. But (by the preceding Proposition) the Sum of all the Quantities $P \times OQ + P' \times OQ' +$ *&c.* is equal to $OG \times$ by the Content of the Body. Therefore OC is likewise $= \frac{P \times OP^2 + P' \times OP'^2 + \text{\&c. \&c.}}{OG \times \textit{Body}}$.

The same otherwise.

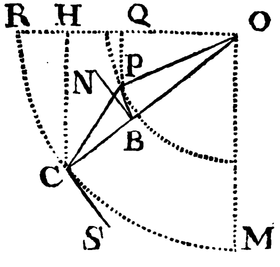
Since the Force of the Particle P, in the perpendicular Direction NB, is defined by $P \times \frac{OP^2}{OB}$, or its Equal, $P \times OQ$, the Sum of all the Quantities $P \times OQ$, $P' \times OQ'$, *&c. &c.* will express the Force which, acting at C perpendicular to OS, is sufficient to stop the Body, without the Center of Suspension O being any way affected: This Sum, therefore, drawn into OC $(= OC \times \overline{P \times OQ + P' \times OQ' + \text{\&c. \&c.}})$ is as the Efficacy of the said Force to turn the Body about the Point O. But the Force of the Particle P, in the Direction BN being $P \times \frac{OP^2}{OB}$, its Efficacy to turn the Body about the same

Point (the contrary way) is as $P \times OP^2$; and consequently the Efficacy of all the Particles as the Sum of all the Quantities $P\times OP^2$, $P'\times OP'^2$ &c. &c. Therefore (Action and Re-action being equal) we have $OC \times \overline{P\times OQ + P'\times OQ' + \text{\&c.}} = P\times OP^2 + P'\times OP'^2 + \text{\&c.}$ *the same as before.*

For the Center of Oscillation, it will be requisite to premise the following

LEMMA.

182. *Suppose two exceeding small Weights* C *and* P, *acting on each other by means of an inflexible Line (or Wire* PC) *to vibrate in a vertical Plane* ROPCM, *about the Center* O; *'tis required to determine how much the Motion of the one is affected by the other.*



Let CH and PQ be perpendicular to the horizontal Line OR; also let PB and CS be perpendicular to OP and OC respectively.

If the Force of Gravity be denoted by Unity, the Forces acting in the Directions CS and PB, whereby the Weights, in their Descent, are accelerated, will, according to the Resolution of Forces, be represented by $\frac{OH}{OC}$ and $\frac{OQ}{OP}$. Moreover, since the Velocities are always in the Ratio of the Radii OC and OP, if the foresaid Forces were to be in that Ratio, or that of P was to become $\frac{OH}{OC} \times \frac{OP}{OC}$, instead of $\frac{OQ}{OP}$. I say, in that Case, it is plain, the Weights would continue their Motion without

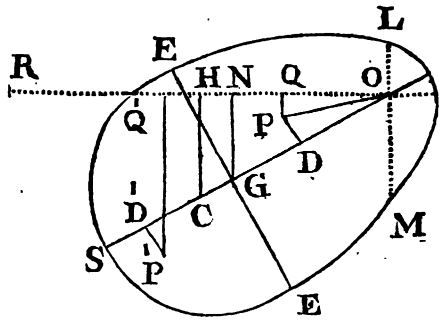
out affecting each other, or acting at all on the Line of Communication PC (or PB). Whence, the Excess of $\frac{OQ}{OP}$ above $\frac{OH}{OC} \times \frac{OP}{OC}$ must be the accelerative Force whereby the Weight P acts upon the Line (or Wire) OC, in the Direction PB; which multiply'd by the Weight P gives $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ for the absolute Force in that Direction: Which therefore, in the perpendicular Direction NB, is $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}} \times \frac{OP}{OB}$; whereof the Part acting upon C, being to the Whole as OB to OC, is truly defined by $P \times \overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$. *Q. E. I.*

If P be supposed to act upon C by means of PC (instead of PB) the Conclusion will be no way different: For, let F (to shorten the Operation) be put to denote the Force $(P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}})$ in the Direction PB, found above, then the Action thereof upon PC (according to the Principles of Mechanics) will be expressed by $F \times \frac{Radius}{Co\text{-}s.\, CPB}$; which therefore in the Direction SC, perpendicular to OC, is $F \times \frac{Radius}{Co\text{-}s.\, CPB} \times \frac{S.\, PCO}{Radius} = \frac{S.\, PCO}{Co\text{-}s.\, CPB} = \frac{S.\, PCO}{S.\, OPC} = F \times \frac{OP}{OC}$, *the very same as before.*

PROPOSITION III.

183. *To determine the Center of Oscillation of a Body.*

The Center of Oscillation is that Point, in the Axis (or Line) of Suspension of a vibrating Body, into which if the whole Body was contracted, the angular Velocity and the Time of Vibration would remain unaltered.



Let LMS be a Section of the Body by a Plane, perpendicular to the Horizon and the Axis of Motion, passing thro' the Center of Gravity G and the Point of Suspension O; and suppose all the Particles of the Body to be transferred to this Section, in such Places of it, as they would be projected into *(orthographically)* by Perpendiculars falling thereon. (Which Supposition will no way affect the Conclusion, the angular Motion continuing the same.) Moreover let C be the Center of Oscillation, or that Point in the Axis OS where a Particle of Matter (or a small Weight) may be placed so as to be neither accelerated nor retarded by the Action of the other Particles of Matter situate in the Plane. Then, if, from C and any other Point P in the Plane LMS, two Perpendiculars CH and PQ be let fall upon the horizontal Line OR, the Force of a Particle (or Weight) at P to accelerate the Weight at C, will *(according to the foregoing Lemma)* be represented by P ×

$\overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$: Which, supposing GN perpendicular to OR, will also be expressed by P $\times$ $\overline{\frac{OQ}{OC} - \frac{ON}{OG} \times \frac{OP^2}{OC^2}}$, or its Equal P $\times$ $\frac{OQ \times OG \times OC - ON \times OP^2}{OC^3}$. In the very same manner the Force of any other Particle $\acute{P}$ will be represented by $\acute{P} \times \frac{O\acute{Q} \times OG \times OC - ON \times O\acute{P}^2}{OC^3}$ *&c. &c.*

Therefore the Forces of all the Particles destroying each other *(by Hypothesis)* the Sum of all the Quantities $P \times \overline{OG \times OQ \times OC - ON \times OP^2}$ $+ \acute{P} \times \overline{OG \times O\acute{Q} \times OC - ON \times O\acute{P}^2}$ *&c. &c.* must be equal to nothing: Whence $P \times OG \times OQ \times OC + \acute{P} \times OG \times O\acute{Q} \times OC$ *&c. &c.* $= P \times ON \times OP^2 + \acute{P} \times ON \times O\acute{P}^2$ *&c. &c.* and consequently $OC = \frac{ON}{OG} \times \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.}{P \times OQ + \acute{P} \times O\acute{Q} + \&c.}$ But *(by Art.* 171. and 172.) the Sum of all the Quantities $P \times OQ + \acute{P} \times O\acute{Q}$ *&c.* is equal to the Content of the Body multiply'd by the Distance (ON) of the Center of Gravity G from the Line LM (perpendicular to OC); whence OC is also $= \frac{ON}{OG} \times$

$$\frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 \ \&c. \&c.}{ON \times Body} = \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 \ \&c. \&c.}{OG \times Body}$$

Which Expression continuing the same in all Inclinations

tions of the Axis OS, the Point C, thus determined is a fixed Point, agreeable to the Definition; and appears to be the same with the Center of Percussion; *see Art.* 181.

COROLLARY.

184. If PD, P'D' &c. be perpendicular to OS, the Numerator of the Fraction found above, will become $P\times\overline{OG^2+GP^2-2OG\times GD}+P'\times\overline{OG^2+GP'^2+2OG\times GD'}$ + &c. &c. (since $OP^2=OG^2+GP^2-2OG\times GD$ &c.) Which, because all the Quantities $P\times-2OG\times GD+P'\times 2OG\times GD'$ &c. or $P\times-GD+P'\times GD'$ &c. (by the Nature of the Center of Gravity) destroy one another, will be *barely* $=P\times\overline{OG^2+GP^2}+P'\times\overline{OG^2+GP'^2}$ + &c. &c. $=\overline{P+P'+\&c.}\times OG^2+P\times GP^2+P'\times GP'^2$ + &c. $=$ *Mass* $\times OG^2+P\times GP^2+P'\times GP'^2$ + &c. Whence it is evident that OC is, also,

$$\left(=\frac{\textit{Mass}\times OG^2,+P\times GP^2+P'\times GP'^2+\&c.\ \&c.}{\textit{Mass}\times OG}\right)$$

$$=OG+\frac{P\times GP^2+P'\times GP'^2+\&c.}{\textit{Mass}\times OG}$$; and consequently

$$CG=\frac{P\times GP^2+P'\times GP'^2+\&c.\ \&c.}{\textit{Mass}\times OG}$$

Whence it appears that, *if a Body be turn'd about its Center of Gravity, in a Direction perpendicular to the Axis of the Motion, the Place of the Center of Oscillation will remain unalter'd*; because the Quantities $P\times GP^2$, $P'\times GP'^2$ are no way affected by such a Motion of the Body.

It

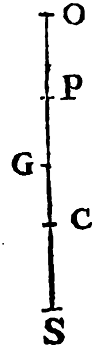
It alſo appears *that the Diſtance of the Center of Gravity from that of Oſcillation* (if the Plane of the Body's Motion remains unalter'd) *will be reciprocally as the Diſtance of the former from the Point of Suſpenſion. Therefore, if that Diſtance be found when the Point of Suſpenſion is in the Vertex, or ſo poſited, that the Operation may become the moſt ſimple, the Value thereof in any other propoſed Poſition of that Point will likewiſe be given, by one ſingle Proportion.*

185. But now, to ſhew how theſe Concluſions may be reduced to Practice, we muſt firſt of all obſerve, that *the Product of any Particle of the Body by the Square of its Diſtance from the Axis of Motion is* (here) *called the Force thereof* (its Efficacy to turn the Body about the ſaid Axis being in that Ratio.) According to which, and the firſt general Value of OC, it appears that, *if the Sum of all the Forces be divided by the Product of the Body into the Diſtance of the Center of Gravity from the Point of Suſpenſion, the Quotient thence ariſing will give the Diſtance of the Center of Percuſſion, or Oſcillation from the ſaid Point of Suſpenſion.*

The Manner of computing the Diviſor has been already explained; it remains therefore to ſhew how the Sum of all the Forces in the Numerator may be collected: Which will admit of ſeveral Caſes. Wherein, to avoid a Multiplicity of Words, I ſhall always expreſs the Diſtance of the Center of Gravity from the Point of Suſpenſion by g, and the Diſtance of the Center of Percuſſion, or Oſcillation, from the ſame Point, by C.

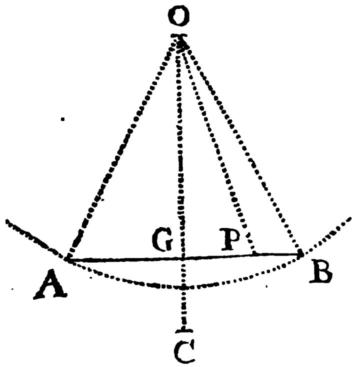
186. Caſe 1. *Let* OS *be a Line ſuſpended at one of its Extremes.*

Then, if the Part OP (conſidered as variable) be denoted by x, the Force of $\dot{x}$ Particles, at P, will *(as above)* be defined by $\dot{x} \times x^2$: Whoſe Fluent ($\frac{1}{3}x^3$) therefore expreſſes the Force of all the Particles in OP (or the Sum of all the Products, under each Particle, and the Square of its Diſtance from O the Point of Suſpenſion. This Quantity therefore (when x becomes



comes $=$ OS) being divided by $OS\times\frac{1}{2}OS$ (according to the foregoing Rule or Observation) we get $\left(\frac{\frac{1}{3}OS^3}{\frac{1}{2}OS^2}=\right)\frac{2}{3}OS$ for the Value of C, the true Distance of the Center of Oscillation (or Percussion) from the Point of Suspension.

187. Case 2. *Let* AB *be a Line, vibrating in a vertical Plane, having its two Extrem s* A *and* B *equally distant from the Point of Suspension* O.

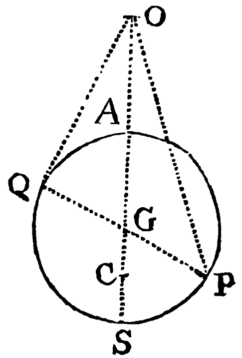


If OG (perpendicular to AB) be put $=a$, and $GP=x$, the Force of $\dot{x}$ Particles at P, will be den ted by $\dot{x}\times\overline{a^2+x^2}=\dot{x}\times OP^2$ *: Whose Fluent, divided by ax (or $PG\times OG$) gives $\left(\frac{a^2x+\frac{1}{3}x^3}{ax}\right)a+\frac{x^2}{3a}=OG+\frac{BG^2}{3OG}=C$, when x becomes $=$ GB.

* Art. 185.

188. Case 3. *Let* APSQ *be a Circle, vibrating in a vertical Plane.* Let PQ be any Diameter thereof; then OP^2+OQ^2 being $=2OG^2+2PG^2$, the Sum of the Forces of two Particles at P and Q, (putting OG $=a$, and $AG=r$) will be $=\overline{a^2+r^2}\times 2$; whence it is evident that the Sum of the Forces of all the Particles, in the whole Periphery, will be expressed by *their Number* $\times\overline{a^2+r^2}$, or by $\overline{a^2+r^2}\times$ *Periph.* APSQ: Which, if

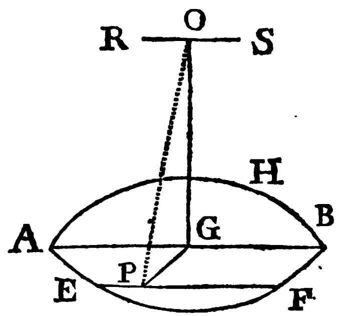
if p be put $= 3.141$ *&c.* will be $= \overline{a^2 + r^2} \times 2pr = 2pa^2r + 2pr^3$. Hence the Force of the Circle itself is also given, being $=$ Fluent of $\overline{2pa^2r + 2pr^3} \times \dot{r} = pa^2r^2 + \frac{1}{2}pr^4 = \overline{a^2 + \frac{1}{2}r^2} \times$ *Circle* APSQ. Now, if the two Expressions thus found be divided by $a \times$ *Periph.* APSQ, and $a \times$ *Circle* APSQ respectively *, we shall have $a + \frac{r^2}{a}$ and $a + \frac{r^2}{2a}$, for the two corresponding Values of C.



* Art. 185.

189. Case 4. *Let* AHBE *be a Circle having its Plane (always) perpendicular to the Axis of Suspension* OG.

Let AGB be that Diameter of the Circle which is parallel to the Axis of Motion RS; and let EF be any Chord parallel to AB and RS; whose Distance, GP, from the Center of the Circle, let be denoted by x; putting OG $= a$, and AG $= r$:

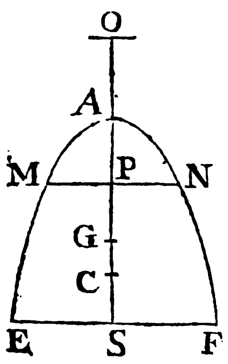


Then, by the Nature of the Circle, EF $= 2\sqrt{r^2 - x^2}$; whose Distance OP, from the Axis of Motion RS, is also given $= \sqrt{a^2 + x^2}$. Hence it appears that the Force of all the Particles in the Line EF (*defined in Art.* 185.) will be represented by $\overline{a^2 + x^2} \times 2\sqrt{r^2 - x^2}$. Therefore $\dot{x} \times \overline{a^2 + x^2} \times 2\sqrt{r^2 - x^2}$ is the Fluxion of the Force of the Plane ABFE; whose Fluent (when $x = r$)

$x = r$) is $= \overline{a^2 + \frac{1}{4}r^2} \times$ *Area AEFBG*; which, if p be put for the Area of the Circle whose Radius is Unity, will be $= \overline{a^2 + \frac{1}{4}r^2} \times \frac{1}{2}pr^2$; whereof the Double ($pa^2r^2 + \frac{1}{4}pr^4$) is the Force of the whole Circle AEFH: whose Fluxion $2parr\dot{r} + pr^3\dot{r}$ (supposing r variable) being divided by $\dot{r}$, we likewise get $2pa^2r + pr^3$ ($= \overline{a^2 + \frac{1}{2}r^2} \times$ *Periph. AEFH*) for the Force of the Periphery AEFH. But the Center of Gravity, whether we regard the Circle itself or its Periphery, is in the Center of the Circle; therefore the Distance of the Center of Oscillation from the Point of Suspension, will in these two Cases be exhibited by $a + \frac{r^2}{4a}$ and $a + \frac{r^2}{2a}$ respectively.

If the Circle, instead of being perpendicular to GO, either coincides, or makes a given Angle with it, the Value of C will come out exactly the same; provided the Diameter AB still continues parallel to the Axis of Motion RS: This appears from *Art.* 184. and may be, otherwise, very easily demonstrated.

190. Case 5. *Let the Figure proposed be a Curve* AEF, *moving (flat-ways, as it were) so that the Plane described by the Axis* OAS *may be perpendicular to that of the Curve.*



Here, putting AP$=x$, PN$=y$, AN$=z$, OA$=d$, OG$=g$, and AG$=a$, the Force of the Particles in MN will be defined by $2y \times \overline{d+x}^2$. Therefore the Fluent of $2y\dot{x} \times \overline{d+x}^2$ will be as the whole Force of the Plane NAM (or AEF, when $x =$ AS) and consequently $C = \frac{Flu.\ \overline{d+x}^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, therefore

fore, when the Point of Sufpenfion is in the Vertex A, will become $C = \frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}$. Let this Value be denoted by v; then, the Diftance of the Centers of Gravity and Ofcillation being $v-a$, we have (by *Art.* 184.) $g : a :: v-a : \left(\frac{a \times \overline{v-a}}{g}\right)$ the Diftance of the fame Centers, when the Point of Sufpenfion is at O, and confequently C, in that Cafe, $= g + \frac{a \times \overline{v-a}}{g}$: Which Form will be found more commodious than the foregoing in moft Cafes.

After the fame Manner the Value of C, with refpect of the Arch AEF, will appear to be $= \frac{Flu.\ \overline{d+x}^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$ $= g + \frac{a \times \overline{v-a}}{g}$, fuppofing $v = \frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}$.

It may not be improper to give an Example or two of the Ufe of the foregoing Theorems.

191. Let therefore EAF be, firft, confider'd as an ifofceles Triangle: In which Cafe AP (x) and PN (y) being in a conftant Ratio, we have $y = \frac{bx}{c}$ (fuppofing SF$=b$ and AS$=c$.) Hence $C\ \left(= \frac{Flu.\ \overline{d+x}^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}\right)$ $= \frac{Flu.\ d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x}}{Flu.\ dx\dot{x} + x^2\dot{x}} = \frac{\frac{1}{2}d^2 + \frac{2}{3}dx + \frac{1}{4}x^2}{\frac{1}{2}d + \frac{1}{3}x} =$ $\frac{6d^2 + 8dx + 3x^2}{6d + 4x}$: Or (according to the fecond Form) becaufe $v\ \left(\frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right) = \frac{3x}{4}$, and a is known to be

* Art. 175. be $= \frac{2x}{3}$ *, we have $C\ (= g + \frac{a \times \overline{v - a}}{g}) = g + \frac{x^2}{18g}$, where $g\ (= d + a) = d + \frac{2}{3}x$.

Again, becaufe $\dot{z}$ and $\dot{x}$ are in a conftant Ratio, we alfo have $\frac{Flu.\ \overline{d+x}|^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}} = \frac{Flu.\ \overline{d+x}|^2 \times \dot{x}}{Flu.\ \overline{d+x} \times \dot{x}} = \frac{d^2 + dx + \frac{1}{3}x^2}{d + \frac{1}{2}x}$; whence the Center of Ofcillation of the Lines EH and AF is given.

192. For a fecond Example, let EAF be fuppofed a Parabola of any Kind, whofe Equation is $y = \frac{x^n}{c^{n-1}}$: Then (*according to Form* 2.) we fhall firft have $v\ (= \frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}) = \frac{Flu.\ x^{n+2}\dot{x}}{Flu.\ x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3}$: Whence,

* Art. 176. a being $= \frac{\overline{n+1} \times x}{n+2}$ *, we alfo get $C\ (= g + \frac{a \times \overline{v-a}}{g})$ $= g + \frac{\overline{n+1} \times x^2}{\overline{n+2}|^2 \times \overline{n+3} \times g}$; where $g = d + \frac{\overline{n+1} \times x}{n+2}$.

But, with refpect to the Arch of the Curve, $v\ (= \frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}})$ is $= \frac{Flu.\ x^2\dot{x}\sqrt{c^{2n-2} + nnx^{2n-2}}}{Flu.\ x\dot{x}\sqrt{c^{2n-2} + nnx^{2n-2}}}$: From which Value (found by infinite Series, and even with-

* Art. 138. out in fome Cafes *) that of C will alfo be given.

193. Cafe 6. *Let the propofed Figure be a Curve vibrating (edge-ways) fo that the Motion of the Axis may be in the Plane of the Curve.*

Then (*by Cafe* 2.) the Force of all the Particles in the Line PN (*fee the preceding Figure*) being defined by $OP^2 \times PN + \frac{1}{3}PN^3$, or $\overline{a+x}|^2 \times y + \frac{1}{3}y^3$ (retaining the No-

Notation above) we have $C = \frac{Flu.\ \overline{d+x}^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, when the Point of Suspension is in the Vertex A, will become $\frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}$: Let this (when found) be denoted by v; then, it appears from the preceding Case, that the general Value of C will, *also*, be represented by $g + \frac{a \times \overline{v-a}}{g}$.

In the same manner the Value of C, with respect to the Arch EAF, will be expounded by $\frac{Flu.\ \overline{\overline{d+x}^2 + y^2} \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$, or by $g + \frac{a \times \overline{v-a}}{g}$, supposing $v = \frac{Flu.\ \overline{x^2 + y^2} \times \dot{z}}{Flu.\ x\dot{z}}$.

194. Example. *Let the Equation of the given Curve be* $y = \frac{x^n}{c^{n-1}}$: Then v $\left(= \frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}\right) =$

$$\frac{Flu.\ c^{1-n}x^{n+2}\dot{x} + \frac{1}{3}c^{3-3n}x^{3n}\dot{x}}{Flu.\ c^{1-n}x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3} +$$

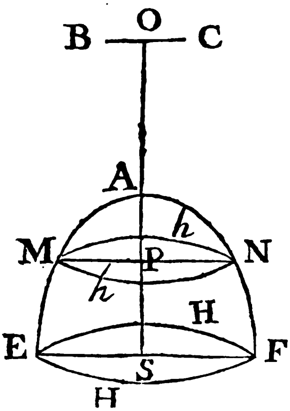
$$\frac{\frac{1}{3}c^{2-2n}x^{3n+1} \times \overline{n+2}}{\overline{3n+1} \times x^{n+2}} = \frac{\overline{n+2} \times x}{n+3} + \frac{\overline{n+2} \times c^{2-2n} \times x^{2n-1}}{3 \times \overline{3n+1}}$$

$= \frac{n+2}{n+3} \times x + \frac{\overline{n+2}}{3 \times \overline{3n+1}} \times \frac{y^2}{x}$: From which the Value of C is also given; and from whence it appears, that if n be expounded by 0, v will become $= \frac{2x}{3} + \frac{2y^2}{3x} = \frac{2}{3} \times \frac{x^2 + y^2}{y}$; in which Case the Figure will degenerate to a Rectangle: But, if n be interpreted by 1, the Figure EAF will then be an isosceles

Triangle,

Triangle, and $v=\frac{3x}{4}+\frac{y^2}{4x}$: Lastly, if n be taken $=\frac{1}{2}$, the Curve will be the common Parabola, and $v=\frac{5x}{7}+\frac{c}{3}$.

195. Case 7. *Let the Figure* AEFH *be a Solid generated by the Rotation of a Curve* EAF *about its Axis* AS; *having its Base* HH *parallel to the Axis of Motion* BOC.



It appears, *from Case* 4. that the Force of all the Particles in the circular Section *bb* (parallel to HH) will be expressed by $\overline{OP^2+\frac{1}{4}PN^2}\times$ *Circle bb*, or $\overline{OP^2\times PN^2+\frac{1}{4}PN^4}\times p$ (p being $=3.1415$ &c.) which, in algebraic Terms, is $\overline{\overline{d+x}^2\times y^2+\frac{1}{4}y^4}\times p$. Hence we have $C=$

$$\frac{Flu.\ \overline{\overline{d+x}^2\times y^2+\frac{1}{4}y^4}\times p\dot{x}}{Flu.\ \overline{d+x}\times py^2\dot{x}} * = \frac{Flu.\ \overline{d+x}^2\times y^2\dot{x}+\frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x}\times y^2\dot{x}}.$$

* Art. 185.

Which, therefore, when the Point of Suspension is in the Vertex A, becomes $\frac{Flu.\ y^2x^2\dot{x}+\frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}}=v$; and consequently $C=g+\frac{a\times\overline{v-a}}{g}$, as in the preceding Cases.

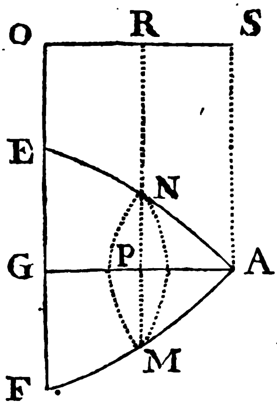
But, with regard to the Superficies of the Solid; it is found, *in Case* 4. that the Force of the Particles in the Periphery M*b*N*b* is expressed by $\overline{OP^2+\frac{1}{2}PN^2}\times$ *Periph. MbNb* $=\overline{d+x}^2\times 2py+py^3$.

Hence

Hence the Fluent of $\overline{d+x}^2 \times 2py + py^3 \times \dot{z}$, divided by that of $\overline{d+x} \times 2py\dot{z}$ $\left(= \frac{Flu.\ \overline{d+x}^2 \times 2y\dot{z} + y^3\dot{z}}{Flu.\ \overline{d+x} \times 2y\dot{z}}\right)$ will give the true Value of C with respect to the curve Surface E*b*A*b*F. Which, putting $v = \frac{Flu.\ 2yx^2\dot{z} + y^3\dot{z}}{Flu.\ 2yx\dot{z}}$, is likewise expressed by $g + \frac{a \times \overline{v-a}}{g}$.

196. Ex. 1. *Let* EAF *be considered as a Cone*; then, putting AS $= f$, SF $= b$ and AF$=c$, we have $y = \frac{bx}{f}$, $z = \frac{cx}{f}$; and therefore C $\left(= \frac{Flu.\ \overline{d+x}^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}\right)$ $= \frac{20d^2 + 30fd + 12f^2 + 3b^2}{20d + 15f}$, when $x = f$. But, with respect to the convex Superficies, C will be found $= \frac{12d^2 + 16df + 6f^2 + 3b^2}{12d + 8f}$.

197. Ex. 2. *Let* EAF *&c. be considered as a Sphere whose Center is* S, *and Radius* AS$=r$; in which Case, y^2 being $= 2rx - x^2$, we have v $\left(= \frac{Flu.\ y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}}\right)$ $= \frac{Flu.\ r^2x^2\dot{x} + rx^3\dot{x} - \frac{3}{4}x^4\dot{x}}{Flu.\ 2rx^2\dot{x} - x^3\dot{x}} = \frac{\frac{1}{3}r^2 + \frac{1}{4}rx - \frac{3}{20}x^2}{\frac{2}{3}r - \frac{1}{4}x}$, whence C is also given. But, when $x = 2r$ (or the whole Sphere is taken) $v = \frac{7r}{5}$: Therefore a being $=r$, and $g = $ OS, in this Case, we have C $\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{r \times 2r}{5g} = g + \frac{2r^2}{5g}$.



198. Case 8. *Let the Figure proposed be a Solid, as in the preceding Case, but let its Axis* AG *be, here, parallel to the Axis of Motion* ORS.

Then, if RP (OG) be put $= g$, 3,1459 &c. $= p$, AP $= x$ &c. the Force of the Particles in the Circle NM (parallel to EF) will be exhibited by $\overline{g^2 + \frac{1}{2}y^2} \times py^2$, or $pg^2y^2 + \frac{1}{2}py^4$ (*Vid. Case* 3.) Hence we have $C =$

* Art. 185. † Art. 145. $\frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Solid}$ * $= \frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Flu.\ py^2\dot{x}}$ † $=$
$g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}$.

Moreover, with respect to the Superficies; the Force of the Particles in the Periphery of the said Circle MN

* Art. 185. being $2pg^2y + 2py^3$ *, we have, in this Case, $C =$

$$\frac{Flu.\ \overline{2pg^2y + 2py^3} \times \dot{z}}{g \times Superficies.} = \frac{Flu.\ 2pg^2y\dot{z} + 2py^3\dot{z}}{g \times Flu.\ 2py\dot{z}} = g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}}.$$

199. Ex. 1. *Let* EAF *be a Segment of a Sphere, whose Radius is r*; then y^2 being $= 2rx - x^2$, we shall have

$$C\left(g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}\right) = g + \frac{Flu.\ 2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x}}{g \times Flu.\ \overline{2rx\dot{x} - x^2\dot{x}}}$$

$$= g + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{g \times \overline{r - \frac{1}{3}x}} = g + \frac{\overline{20r^2 - 15rx + 3x^2} \times x}{\overline{30r - 10x} \times g}.$$

Which, when x is expounded, either, by r or $2r$, becomes $= g + \frac{2r^2}{5g}$, for the true Value of C, when

either

either the Hemisphere, or whole Sphere, is taken. But, with respect to the Center of Oscillation of the Superficies thereof, we have $\dot{z}$ in this Case $= \frac{r\dot{x}}{\sqrt{2rx - xx}}$* $= \frac{r\dot{x}}{y}$: And therefore $g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}} = g + \frac{Flu.\ \overline{2rx - xx} \times r\dot{x}}{g \times Flu.\ r\dot{x}} = g + \frac{rx - \frac{1}{3}x^2}{g}$: Which, when $x = r$, or $x = 2r$, becomes $g + \frac{2r^2}{3g}$.

* Art. 142.

200. Ex. 2. *Let the Solid* EAF *be a Paraboloid, whose generating Curve is defined by the Equation* $y = \frac{x^n}{c^{n-1}}$:

Then $C = g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}} = g + \frac{Flu.\ \frac{1}{2}x^{4n}\dot{x} \times c^{4-4n}}{g \times Flu.\ x^{2n}\dot{x} \times c^{2-2n}} = g + \frac{\overline{2n+1} \times x^{2n}}{\overline{4n+1} \times 2gc^{2n-2}} = g + \frac{\overline{2n+1} \times y^2}{\overline{4n+1} \times 2g}$. Where, if n be taken $= 0$, the Figure will become a Cylinder, and $C = g + \frac{y^2}{2g}$: But if n be expounded by 1, the Figure will be a Cone, and $C = g + \frac{3y^2}{10g}$. Lastly, if n be taken $= \frac{1}{2}$, the Figure will be the Solid generated from the common Parabola and $C = g + \frac{y^2}{3g}$.

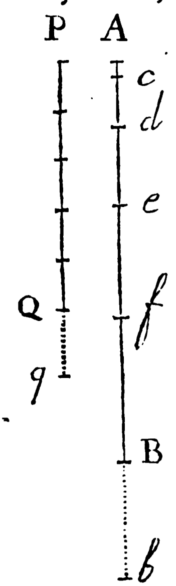
SECTION XII.

Of the Use of FLUXIONS *in determining the Motion of Bodies affected by centripetal Forces.*

PROPOSITION I.

201. *THE Motion, or Velocity, acquired by a Body freely descending from Rest, by the Force of an uniform Gravity, is proportional to the Time of its Descent; and the Space gone over, as the Square of that Time.*

The first Part of the Proposition is almost self-evident: For, since any Motion is proportional to the Force by which it is generated, that generated by the Force of an uniform Gravity must be as the Time of Descent; because the whole Effect of such a Force, acting equally every Instant, is as that Time.



Let, now, the Velocity acquired during a Descent of one Second of Time, be such as would carry the Body uniformly over any Distance b in one Second; and let AB (x) denote the Distance descended in any proposed Time t; which Time let be denoted by PQ; making $Bb = \dot{x}$ and $Qq = \dot{t}$: Then it will be, as $1'' : t :: b : (bt)$ the Distance that would be uniformly described in $1''$, with the Velocity at B: Also $1'' : \dot{t} ::$ the said Distance (bt) to $bt\dot{t} = \dot{x}$ *. By taking the Fluent whereof we get $\frac{1}{2}bt^2$

* Art. 2.

$\frac{1}{2}bt^2 = x$. Therefore the Diſtance deſcended ($\frac{1}{2}bt^2$) is as the Square of the Time. *Q. E. D.*

Otherwiſe, without Fluxions.

Conceive the Time (PQ) of falling thro' AB to be divided into an indefinite Number of very ſmall equal Particles, repreſented, each, by m; and let the Diſtance deſcended in the firſt of them be Ac, in the ſecond cd, in the third de, &c. &c. Then, the Velocity being always as the Time from the Beginning of the Deſcent, it will in the Middle of the firſt of the ſaid Particles be defined by $\frac{1}{2}m$; in the Middle of the ſecond by $1\frac{1}{2}m$; in the Middle of the third by $2\frac{1}{2}m$, &c. &c. But, ſince the Velocity at the Middle of any Particle of Time, is a Mean between thoſe at the two Extremes, or betwixt any other two equally remote from it, the correſponding Particle of the Diſtance AB may, therefore, be conſidered as deſcribed by that mean Velocity. And ſo, the Spaces Ac, cd, de, ef, &c. deſcribed in equal Times, being reſpectively as the ſaid mean Celerities $\frac{1}{2}m$, $1\frac{1}{2}m$, $2\frac{1}{2}m$, $3\frac{1}{2}m$, &c. it follows, by Addition, that the Diſtances Ac, Ad, Ae, Af, &c. gone over from the Beginning, are to one another as $\frac{m}{2}$, $\frac{4m}{2}$, $\frac{9m}{2}$, $\frac{16m}{2}$, &c. or 1, 4, 9, 16, 25, &c. that is, as the Squares of the Times. *Q. E. D.*

COROLLARY I.

202. Since the Diſtance that might be uniformly run over in one Second, with the Velocity at B, is expreſſed by bt, the Diſtance that might be deſcribed with the ſame Velocity in the Time t will therefore be expreſſed by $bt \times t$, or bt^2: Whence it appears, that the Space AB ($\frac{1}{2}bt^2$) thro' which the Body falls in any given Time t, is juſt the half of that which would be uniformly deſcribed with the Celerity at B, in the ſame Time.

Therefore, ſince it is found from Experiment, that a Body near the Earth's Surface (where the Gravity may be taken as uniform) descends about $16\frac{1}{12}$ Feet in the first Second, it follows that the Value of b (is in this Case) $=2\times16\frac{1}{12}=32\frac{1}{6}$: And consequently the Number of Feet descended in t Seconds, equal to $16\frac{1}{12}\times t^2$.

COROLLARY 2.

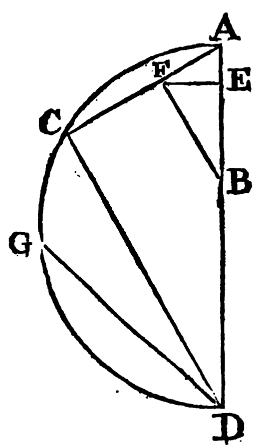
203. It is evident, whatever Force the Body descends by, the Value of b will always be as that Force; since a double Force, in the same time, generates a double Velocity; a treble Force, a treble Velocity, &c. Therefore, seeing our Equation $\frac{1}{2}bt^2=x$, also gives $t=\sqrt{\frac{x}{\frac{1}{2}b}}$, and $b=\frac{x}{\frac{1}{2}t^2}$, it follows,

1. That the Distance descended is, universally, as the Force and the Square of the Time conjunctly.

2. That the Time is always as the Square-root of the Distance applied to the Force.

3. And that the Force is as the Distance apply'd to the Square of the Time —— And it may be further observed, that, whatever is here said with regard to the Time, also holds in the Velocity, being proportional to the Time.

PROPOSITION II.



204. *To determine the Velocity, and Time of Descent, of a Body along an inclined Plane* AC.

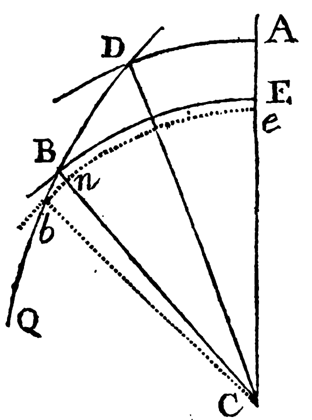
From any Point F, in AC, draw FE perpendicular to the vertical Line AD, and make FB and CD perpendicular to AC, meeting AD in B and D. Because (by the Principles of Mechanics) the Force of Gravity in the Direction CF, whereby the Body is made to descend along the Plane, is to the absolute Force thereof, as AF to AB, AB, or as AC to AD; and since *(by Case* 1. *Art.* 203.) the Distances descended in equal Times are as the Forces, it follows that the Time of Descent thro' AF will be equal to the Time of the perpendicular Descent thro' AB: And consequently the Time of Descent thro' AC equal to that thro' AD; which is given by *Prop.* 1. Moreover, because the Velocities at F and B, acquired in equal Times, are as the Forces, or as AF to AB; and it appears from *Prop.* 1. that the Velocity at E is to that at B, as $\sqrt{AE} : \sqrt{AB}$, or as $\sqrt{AE \times AB}$ (=AF) : $\sqrt{AB \times AB}$ (=AB) it follows, by Equality, that the Celerity at F is equal to to that at E; which is therefore given, by the preceding Proposition. *Q. E. I.*

COROLLARY.

205. Hence the Time of Descent along the Chord AC of a Semi-circle ACD is equal to the Time of Descent along the vertical Diameter AD: And, if the Chord DG be of the same Length with AC (its Inclination to the Horizon being also the same) the Time of Descent along it will also be equal to that along the vertical Diameter.

PROPOSITION III.

206. *If, from two Points* A *and* D, *equally remote from the Center of Attraction* C, *two Bodies move, with equal Celerities, the one along the Right-line* AC, *the other in a Curve-line* DBQ, *their Celerities, at all other equal Distances from the Center, will be equal.*



For, let CB and CE be any two such Distances; let the Arch BE be de-

 scribed,

ſcribed, from the Center C, and alſo *eb*, indefinitely near to it, cutting CB in *n*: Let the centripetal Force at the Diſtance of CB, or CE, be repreſented by *f*, and the Velocity at B, by *v*.

By the Reſolution of Forces, the Efficacy of the Force (*f*) in the Direction B*b*, whereby the Velocity of the Body is accelerated, will be $\frac{Bn}{Bb} \times f$: Alſo the Time of moving over B*b* (being as the Diſtance apply'd to the Velocity) is repreſented by $\frac{Bb}{v}$: Therefore the Increaſe of Velocity, in moving thro' B*b*, being as the Force and Time conjunctly, will be defined by $\frac{Bn}{Bb} \times f \times \frac{Bb}{v}$, or its Equal $\frac{Bn}{v} \times f$. In the ſame Manner, the Velocity at E being denoted by *w*, the Time of falling thro' E*e* will be repreſented by $\frac{Ee}{w}$, and the Velocity generated in that Time by $\frac{Ee}{w} \times f$: Which is to that $\left(\frac{Bn}{v} \times f\right)$ acquired in falling thro' the Arch B*b*, as $\frac{1}{w}$ to $\frac{1}{v}$. Therefore, ſeeing the correſponding Increments of Velocity are always, reciprocally, as the Velocities themſelves, it is manifeſt, if thoſe Velocities are equal, in any two correſponding Poſitions of the Bodies, they will be ſo in all others, being always increaſed alike. But they are equal at A and D by Suppoſition: Therefore, &c. *Q. E. D.*

PROPOSITION IV.

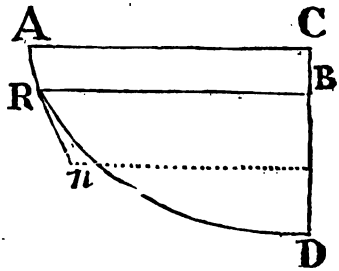
207. *To find the Ratio of the Velocities, and Times of Deſcent, of Bodies, in Curves; the Force of Gravity being conſidered as uniform.*

Let ARD repreſent a Curve of any Kind, along which a Body deſcends, by the Force of its own Gravity

vity from A; let AC, RB, *&c.* be parallel, and CD perpendicular, to the Horizon; moreover, let Rn touch the Curve at R; and let CB $= u$, AR $= w$, and R$n = \dot{w}$ *.

* Art. 135.

Since the Points B and R (as well as C and A) may be looked upon as equally remote from the Earth's Center (to which the Gravitation tends), the Velocity acquired in descending thro' the Arch AR will (*by the last Proposition*) be



equal to that acquired by falling freely thro' the Right-line CB; which last Velocity (*by Prop.* 1.) is always as $\sqrt{CB}$ (or $u^{\frac{1}{2}}$). Therefore the Celerity, whether the Body moves in a Right-line, or a Curve, is always in the subduplicate Ratio of the perpendicular Descent; and so, the Time in which Rn ($\dot{w}$) would be uniformly described, with that Celerity, will be universally as $\frac{\dot{w}}{u^{\frac{1}{2}}}$; whose Fluent is as the Time of falling thro' AR. *Q. E. I.*

EXAMPLE.

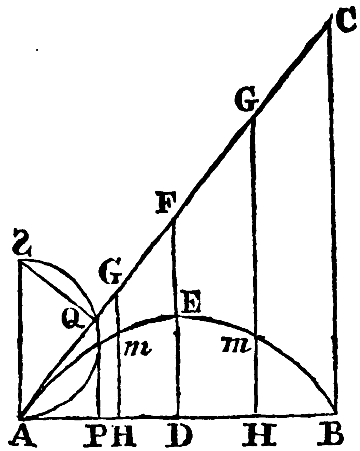
208. Let the Curve ARD be any Portion of the common Cycloid; whereof the Vertex is D and Axis DC; and whose Nature (putting DC$=c$, and the Ray of Curvature at D $= a$) is defined by the Equation $2a \times DB = DR^2$. Here, we have DR $(=\sqrt{2a}\times\sqrt{DB})$ $= \sqrt{2a} \times \overline{c-u}^{\frac{1}{2}}$; whose Fluxion $-\sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\overline{c-u}^{\frac{1}{2}}}$, with a contrary Sign, is the Value of Rn or $\dot{w}$;

and

therefore $\frac{\dot{w}}{u^{\frac{1}{2}}} = \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\sqrt{cu - uu}}$: Whose Fluent, at the loweſt Point D, where u becomes $= c$, will (*by Art.* 142.) be equal to $\sqrt{2a}$ multiply'd by $\left(\frac{3.14159 \&c.}{2}\right)$ half the Meaſure of the Periphery of the Circle whoſe Diameter is Unity. Which Fluent (and conſequently the Time of Deſcent) will therefore continue the ſame, let the Arch DA be what it will.

PROPOSITION V.

209. *To determine the Paths of Projectiles near the Earth's Surface; (neglecting the Reſiſtance of the Atmoſphere.)*



Let a Body be projected from the Point A, in the Direction of the Line AC, with a Velocity ſufficient to carry it uniformly over the Diſtance d in the Time t; and let the Space thro' which it would freely deſcend, by its own Gravity, in that time, be denoted by b; alſo let the Sine of the Angle of Elevation BAC (to the Radius r) be put $= s$, its Co-ſine $= c$, and the Diſtance of the Point A from the Ordinate Hm (conſidered as moving parallel to itſelf along with the Body) $= x$; then, *by Trig.* HG (perpendicular to AB) will be $= \frac{sx}{c}$, and AG $= \frac{rx}{c}$.

Becauſe the Projectile is turned aſide, continually, from a rectilinear Path, by the Earth's Attraction, it muſt

muſt deſcribe a Curve-line A*m*E*m*B, to which AC is a Tangent at the Point A: But that Attraction, acting always in a Direction (H*m*) perpendicular to the Horizon, can have no Effect upon that Part of the Velocity with which the Body approaches the Line BC, parallel to H*m*; therefore the Right-line HG (in which the Body is always found) will continue to move uniformly towards BC, the ſame as if Gravity was not to act; and the Diſtance G*m* deſcended from the Tangent AC, by means of the Attraction, will be the very ſame as if the Body was to deſcend from Reſt along the Line GH. This being premiſed, it is evident, that as d : AG $\left(\frac{rx}{c}\right) :: t : \left(\frac{rx}{cd} \times t\right)$ the Time of deſcribing A*m*; and, as $t^2 : \frac{r^2x^2}{c^2d^2} \times t^2 :: b : \left(\frac{br^2x^2}{c^2d^2}\right)$ the Space (G*m*) thro' which a Body would freely deſcend in that Time (*by Prop.* 1.)

Hence $\frac{sx}{c} - \frac{br^2x^2}{c^2d^2}$, or $\frac{csd^2x - br^2x^2}{c^2d^2}$ is a general Value for the Ordinate H*m*: By putting which $= 0$, we get $x = \frac{csd^2}{br^2} =$ AB $=$ the Amplitude of the Projection. But, by putting its Fluxion equal to nothing, we have $x = \frac{csd^2}{2br^2}$; which ſubſtituted for x in the Value of H*m*; gives $\frac{s^2d^2}{4br^2}$ for the Altitude DE of the Projection. *Q. E. I.*

COROLLARY.

210. If another Body be projected, with the ſame Celerity, in the vertical Direction AS; then, s becoming $= r$, the Altitude of that Projection $\left(\frac{s^2d^2}{4br^2}\right)$ will be-

come

come $\frac{d^2}{4b}$ = AS; which call h, and let this Value be ſubſtituted in thoſe of AB and DE, and they will become $\frac{4hcs}{r^2}$ and $\frac{hs^2}{r^2}$ reſpectively.

Hence, if from the Point Q where the Line of Direction AC cuts a Semi-circle deſcribed upon AS, the Lines SQ and QP be drawn, the latter perpendicular to AB, the Triangles ASQ and AQP being ſimilar, we ſhall have

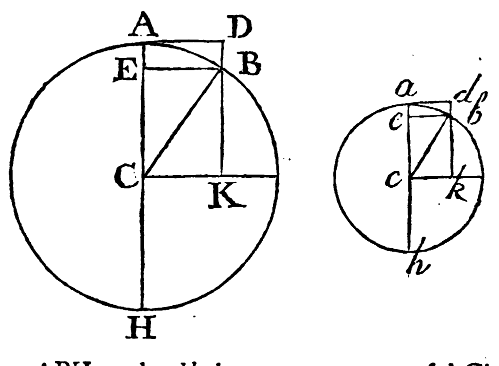
$$r : s :: h\ (\text{AS}) : \frac{sh}{r} = \text{AQ}$$

$$r : s :: \frac{sh}{r}\ (\text{AQ}) : \frac{s^2h}{r^2} = \text{PQ} = \text{DE}$$

$$r : c :: \frac{sh}{r}\ (\text{AQ}) : \frac{sch}{r^2} = \text{AP} = \tfrac{1}{4}\ \text{AB}$$

PROPOSITION VI.

211. *To determine the Ratio of the Forces, whereby Bodies, tending to the Centers of given Circles, are made to revolve in the Peripheries thereof.*



Let ABH and *abh* be any two propoſed Circles, whereof let AB and *ab* be ſimilar Arcs; in which, let the

the Velocities of the revolving Bodies be reſpectively as V to v; make DBK and dbk parallel to the Radii AC and ac, putting AC$=R$, $ac=r$, and the Ratio of the centripetal Force in ABH to that in abh, as F to f.

It is plain, becauſe the Angles ABD and abd are equal, that the Velocities at B and b, in the Directions BK and bk, with which the Bodies recede from the Tangents AD and ad, are to each other as the abſolute Celerities V and v *. But thoſe Velocities, being the Effects of the centripetal Forces acting in correſponding, ſimilar, Directions during the Times of deſcribing AB and ab, will therefore be as the Forces themſelves when the Times are equal; but when unequal, as the Forces and Times conjunctly. Therefore, the Times being univerſally as $\frac{AB}{V}$ to $\frac{ab}{v}$, or as $\frac{R}{V}$ to $\frac{r}{v}$ (becauſe the Arcs AB and ab are ſimilar) we have, as $F\times\frac{R}{V}:f\times\frac{r}{v}::V:v$; whence (multiplying the Antecedents by $\frac{V}{R}$ and the Conſequents by $\frac{v}{r}$) it will be, as $F:f::\frac{V^2}{R}:\frac{v^2}{r}$: Therefore the Forces are as the Squares of the Velocities directly, and as the Radii inverſely.

* Art. 35.

Otherwiſe.

Let the indefinitely little Arch AB be the Diſtance that the Body moves over in a given, or conſtant Particle of Time; and let the centripetal Force at B be meaſured by twice the Subtenſe or Space AE thro' which the Body is drawn from the Tangent AD in that Time †.

Then,

† *The Velocity which any Force, uniformly continued, is capable of generating, in a given Body, in a given Time, is the proper Meaſure of the Intenſity of that Force* *. *But this Velocity is itſelf meaſured by the Space the Body would move uniformly*

* Art. 203.

Then, by the Nature of the Circle, $AB^2 = AH \times AE = AC \times 2AE$, and consequently $2AE = \frac{AB^2}{AC}$: Therefore, the Force is as the Square of the Velocity apply'd to the Radius of the Circle *(as before.)*

COROLLARY I.

212. Because, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$

$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

COROLLARY II.

213. If the Ratio of the periodic Times be denoted by that of P to p; then the Ratio of the Velocities V, v being as $\frac{R}{P}$ to $\frac{r}{p}$, we shall have, by Equality, $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence also

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$

$$R : r :: FP^2 : fp^2.$$

formly over in a given Time; which Space is always the double of that thro' which the Body would freely descend, from Rest, in the same time ‡. *Therefore* 2AE *is the proper Measure of the centripetal Force, according as we have assumed it.—— 'Tis true, when the Forces to be compared are all computed in the same Manner, from the Nascent, or indefinitely small Subtenses of contemporaneous Arcs, it matters not whether we consider those Subtenses, or their Doubles, as the Measures of the Forces, the Ratio being the same in both Cases. But when the Forces so found are to be compared with others derived from a fluxional Calculus, it is absolutely necessary to take the double Subtense for the Measure of the Force.—— This Note is inserted, that the Learner may avoid the Errors, which some very considerable Mathematicians have fallen into by not properly attending to this Particular.*

‡ Art. 202.

Co-

Corollary III.

214. If the Meaſure of the Force, or the Velocity that might be uniformly generated in a given Time (1) be expounded by any Power a^n of the Radius AC (a); then the Diſtance thro' which a Body would freely deſcend in the ſame Time, by that Force, uniformly continued, will be expreſſed by $\frac{1}{2}a^n$ *. Therefore, the Diſtances deſcended, by means of the ſame Force, uniformly continued, being as the Squares of the Times ‡, it is evident, if the Time of moving thro' AB be denoted by t, that the Diſtance AE deſcended in that Time, will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2}a^n$: And ſo we ſhall have AB $(\sqrt{2AE \times AC}) = \frac{t}{1} \times a^{\frac{n+1}{2}}$; which being the Diſtance deſcribed by the revolving Body in the Time t, it follows that the Space gone over in the given Time (1) will be $a^{\frac{n+1}{2}}$: Which, therefore, is the true Meaſure of the Celerity in this Caſe. The ſame Concluſion might have been derived in much fewer Words from *Corol.* 1. but, as a thorough underſtanding hereof is abſolutely neceſſary in what follows hereafter, I have endeavoured to make it as plain as poſſible.

* Art. 202.

‡ Art. 201.

Corollary IV.

215. Hence the Time of Revolution is alſo derived; for it will be as $a^{\frac{n+1}{2}}$: 3.14159 &c. $\times\, 2a$ (the whole Periphery) :: 1 : $\frac{3.14 \text{ \&c. } \times 2a}{a^{\frac{n+1}{2}}}$ or 3.14159 &c. $\times\, 2a^{\frac{1-n}{2}}$, the true Meaſure of the periodic Time.

Co-

COROLLARY V.

216. Therefore, if n be expounded by 1, 0, -1, -2 and -3 successively, then the Velocity corresponding will be as a, $a^{\frac{1}{2}}$, 1, $a^{-\frac{1}{2}}$, and a^{-1}; and the Time of Revolution, as 1, $a^{\frac{1}{2}}$; a, $a^{\frac{3}{2}}$ and a^2 respectively.

SCHOLIUM.

217. From the preceding Proposition, and its subsequent Corollaries, the Velocity and periodic Time of a Body revolving in a Circle at any given Distance from the Earth's Center, by means of its own Gravity, may be deduced: For let d be put for the Space thro' which a heavy Body, at the Surface of the Earth, descends in the first Second of Time, then $2d$ will be the Measure of the Force of Gravity at the Surface: And therefore, the Radius of the Earth being denoted by r, the Velocity, *per* Second, in a Circle at its Surface, will be $\sqrt{2rd}$; and the Time of Revolution $= \frac{3.14159 \text{ \&c.} \times 2r}{\sqrt{2rd}}$ $= 3.14159 \text{ \&c.} \times \sqrt{\frac{2r}{d}}$ (*Seconds*); which two Expressions, because r is $= 21000000$ Feet and $d = 16\frac{1}{12}$ will therefore be nearly equal to 26000 Feet and 5075 *Seconds*, respectively. Let R be now put for the Radius of any other Circle described by a Projectile about the Earth's Center: Then, because the Force of Gravitation above the Surface is known to vary according to the Square of the Distance inversly, we have (*by Case* 4. *Corol.* 5.) $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} ::$ (26000 F) the Velocity (*per* Second) at the Surface, to $26000 \times \sqrt{\frac{r}{R}}$, the Velocity

locity in the Circle whose Radius is R: And $r^{\frac{3}{2}} : R^{\frac{3}{2}}$ :: (5075 S.) the periodic Time at the Surface : to $5075 \times \sqrt{\frac{R^3}{r^3}}$, the Time of Revolution in the Circle R. Which, if R be assumed equal to (60r) the Distance of the Moon from the Earth, will give 2360000 S., or 27.3 D nearly, for the periodic Time at that Distance.

In like sort the Ratio of the Forces of Gravitation of the Moon, towards the Sun and Earth, may be computed. For, the centrifugal Forces in Circles, being universally as the Radii apply'd to the Squares of the Times of Revolution, it will be as $\left(\frac{81000000}{1}\right)$ the Semi-diameter of the *Magnus Orbis* divided by the Square of one Year (the periodic Time of the Earth and Moon about the Sun) is to (240000 × 178) the Distance of the Moon from the Earth divided by $\frac{1}{178}$, the Square of the periodic Time of the Moon about the Earth, so is 1,9 to 1 nearly; and so is the Gravitation of the Moon towards the Sun to her Gravitation towards the Earth.

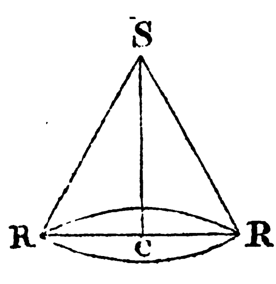
Also, after the same Manner, the centrifugal Force of a Body at the Equator, arising from the Earth's Rotation, is derived. For since it is found above, that 5075 Seconds is the Time of Revolution, when the centrifugal Force would become equal to the Gravity, and it appears (*by Case* 2. *Corol.* 2.) that the Forces, in Circles having the same Radii, are inversly as the Squares of the periodic Times, we therefore have, as $\overline{86160}|^2$ (the Square of the Number of Seconds in (23 H 56 M) one whole Rotation of the Earth) to $\overline{5075}|^2$ (the Square of the Number of Seconds above given) so is the Force of

R Gravity

Gravity (which we will denote by Unity) to $\frac{1}{289}$, the centrifugal Force of a Body at the Equator arising from the Earth's Rotation.

But, to determine, in a more general Manner, the Ratio of the Force of a Body revolving in any given Circle, to its Gravity, we have already given 3.14 &c. x $\sqrt{\frac{2r}{d}}$ for the Time of Revolution at the Surface of the Earth, when the Gravity and centrifugal Force are equal: Therefore, if the Time of Revolution in any Circle whose Radius is a, be denoted by t, it follows, from *Corol.* 2. *last Prop.* that, $\frac{r}{\overline{3.14}|^2 \text{ \&c.} \times \frac{2r}{d}} : \frac{a}{t^2}$ :: the Gravity of the Body : to its centrifugal Force in that Circle; which, therefore, is as Unity to $\frac{\overline{3.14}|^2 \text{ \&c.} \times 2a}{dt^2}$; or as 1 to $1.228 \times \frac{a}{t^2}$ very nearly; where a denotes the Number of Feet in the Radius of the proposed Circle, and t the Number of Seconds in one intire Revolution. So that, if the Length of a Sling, by which a Stone is whirled about, be two Feet, and the Time of Revolution $\frac{1}{2}$ of a Second, the Force by which the Stone endeavours to fly off, will be to its Weight as 9.824 to Unity.

From this general Proportion, the centrifugal Force and periodic Time of a Pendulum describing a conical Surface may likewise be deduced.



For let SR, the Length of the Pendulum, be denoted by g; the Altitude CS of the Cone, by c; the Semi-diameter CR of the Base by a; and the Time of Revolution by t: Then, the Force of Gravity being re-

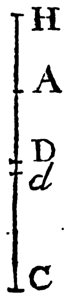
represented by Unity, the Force with which the revolving Body at R, the End of the Pendulum, tends to recede from the Center C, will be defined by $\frac{\overline{3.14 \&c.}|^2 \times 2a}{dt^2}$, as has been already shewn. Therefore, because the Body is retained in the Circle RR by the Action of three different Powers; *i. e.* the centrifugal Force $\left(\frac{\overline{3.14 \&c.}|^2 \times 2a}{dt^2}\right)$ in the Direction CR, the Force of Gravity (1) in a Direction parallel to SC, and the Force of the Thread or Wire RS, compounded of the former two; it follows, from the Principles of Mechanics, that as SC (c) to CR (g), so is the Weight of the Body at R, to the Force with which it acts upon the Thread or Wire RS; and as $1 : \frac{\overline{3.14 \&c.}|^2 \times 2a}{dt^2}$ :: CS (c) : CR (a) : Whence $dt^2 = \overline{3.14 \&c.}|^2 \times 2c$; and $t = 3.14 \&c. \times \sqrt{\frac{2c}{d}} = 1,108 \sqrt{c}$ nearly. Because dt^2, or its Equal $\overline{3.14 \&c.}|^2 \times 2c$, expresses the Space a heavy Body will descend, by its own Gravity, in the Time t *, and since $1^2 : \overline{3.14 \&c.}|^2 :: 2c :$ $\overline{3.14 \&c.}|^2 \times 2c$ $(= dt^2)$ it therefore appears that, as the Square of the Diameter of any Circle, is to the Square of its Periphery, so is twice the perpendicular Altitude of the Cone, to the Distance a heavy Body will freely descend in the Time of one whole Gyration of the Pendulum, let the Base of the Cone and the Length of the Pendulum be what they will.

* Art. 202.

PROPOSITION VII.

218. *To determine the Ratio of the Velocities of Bodies descending, or ascending, in Right-lines, when accelerated, or retarded, by Forces, varying according to a given Law.*

Suppose a Body to move in the Right-line CH, and let the Force whereby it is urged towards C, or H, be be as any variable Quantity F: Moreover, let the Velocity of the Body be represented by v; putting its Distance CD, from the Point $C = x$, and $Dd = \dot{x}$.



Then, since the Time wherein the Space Dd $(\dot{x})$ would be uniformly described, with the Velocity at D, is known to be as $\frac{\dot{x}}{v}$, the Velocity that would be uniformly generated, or destroyed, in that Time by the Force F (being as the Time and Force conjunctly) will consequently be as $\frac{F\dot{x}}{v}$: Which therefore must be equal to, $\pm \dot{v}$, the uniform Increase or Decrease of Celerity in that Time; and consequently $\pm v\dot{v} = F\dot{x}$. From whence, when the Value of F is given in Terms of x, or v, the Value of v will likewise be known. *Q. E. I.*

COROLLARY I.

219. Hence, the Law of the Velocity being given, that of the Force is deduced: For, since $F\dot{x} = \pm v\dot{v}$, it is evident that $F = \pm \frac{v\dot{v}}{\dot{x}}$.

COROLLARY II.

220. Hence, also, the Ratio of the Velocity at D to that whereby a Body might revolve in the Periphery of a Circle about the Center C, at the Distance of CD, will be known: For, if this last Velocity be denoted by w, the Value of F will be rightly expressed by $\frac{w^2}{x}$ *: Whence, by Substitution, we have $\pm v\dot{v} = \frac{w^2\dot{x}}{x}$, or

* Art. 212.

$\pm \dot{v}^2$

$\pm v^2 \times \frac{\dot{v}}{v} = w^2 \times \frac{\dot{x}}{x}$: Whence $w^2 : v^2 : \pm \frac{\dot{v}}{v} : \frac{\dot{x}}{x}$, and consequently $w : v :: \sqrt{\pm \frac{\dot{v}}{v}} : \sqrt{\frac{\dot{x}}{x}}$. Where, as well as above, the Sign of $\dot{v}$ must be taken + or — according as the Body is urged from, or towards the Center C.

PROPOSITION VIII.

221. *Supposing a Body, let go from a given Point* A *with a given Celerity (c) along a Right-line* CH, *to be urged, either way, in that Line, by a Force varying as any Power (n) of the Distance from a given Point* C; *to find, not only, the Relation of the Velocities, and Spaces gone over, but also the Times of Ascent and Descent.*

The Construction of the preceding Problem being retained, F will here be expounded by x^n, and we shall thefore have $\pm v\dot{v}\ (= F\dot{x}) = x^n \dot{x}$; and consequently, by taking the Fluent thereof, $\pm \frac{vv}{2} = \frac{x^{n+1}}{n+1}$; but to correct the Fluent thus found, let x be taken $=$ CA (which we will call a) then v being $= c$, the Fluent in that Circumstance will become $\pm \frac{c^2}{2} = \frac{a^{n+1}}{n+1}$: Therefore the Fluent duly corrected is $\pm \frac{v^2}{2} \mp \frac{c^2}{2} = \frac{x^{n+1} - a^{n+1}}{n+1}$*, or $v^2 \backsim c^2 = \frac{2x^{n+1} \backsim 2a^{n+1}}{n+1}$: Whence v will * Art. 78. come out $= \sqrt{c^2 + \frac{\mp 2a^{n+1} \pm 2x^{n+1}}{n+1}}$: Where the Signs of v and x^{n+1} must be alike, when both Quantities increase, or decrease, at the same time; that is,

 when

* Art. 220. when the Force, from C, is a repulsive one *; but, unlike, when one increases while the other decreases, or the Force, tending to C, is an attractive one. In the former Case we therefore have $v = \sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}$; and, in the latter, $v = \sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$.

The Value of v being thus obtained, let the required Time of moving over the Space AD be now denoted by T; then, since $\dot{T}$ is universally $= \frac{\dot{x}}{v}$, we have $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}}$, or $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}}$ according to the two foresaid Cases respectively: From whence, by finding the Fluent, the Time itself will be known. *Q. E. I.*

COROLLARY.

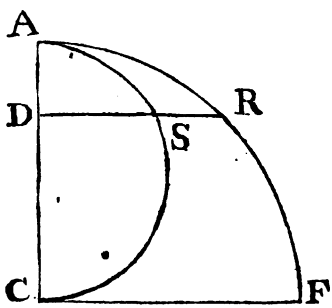
222. If the Body proceeds from Rest at A, c will be $= 0$, and we shall have $\dot{T} = \frac{\overline{1+n}^{\frac{1}{2}} \times \dot{x}}{\sqrt{2x^{n+1} - 2a^{n+1}}}$, or $\dot{T} = \frac{\overline{1+n}^{\frac{1}{2}} \times \dot{x}}{\sqrt{2a^{n+1} - 2x^{n+1}}}$.

SCHOLIUM.

223. Although, the Fluents of the Expressions given above cannot be exhibited, in a general Manner, neither, in finite Terms, nor by means of circular Arcs and Logarithms; yet, in some of the most useful Cases

Cases that occur in Nature, they may be obtained with great Facility.

Thus, if in $\frac{\overline{1+n}|^{\frac{1}{2}}\,\dot{x}}{\sqrt{2a^{n+1}-2x^{n+1}}}$ (expressing the Fluxion of the Time of Descent along AD) n be expounded by 1, 0, -2, and -3 successively, the Fluxion itself will become equal to $\frac{\dot{x}}{\sqrt{a^2-x^2}}$, $\frac{\dot{x}}{\sqrt{2a-2x}}$, $\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, and $\frac{ax\dot{x}}{\sqrt{a^2-x^2}}$ respectively: Whence, if ARF be a Quadrant of a Circle whose Center is C, and ASC a Semi-circle whose Diameter is AC, and DSR be perpendicular to AC; then it will appear,



1°. That, when $n=1$, and $\dot{T}=\frac{\dot{x}}{\sqrt{a^2-x^2}}$, the Velocity $(\sqrt{a^2-x^2})$ at D will be represented by DR, and the Fluent sought by $\frac{AR}{AC}$*. *Art. 142.

2°. That, when $n=0$, and $\dot{T}=\frac{\dot{x}}{\sqrt{2a-2x}}$, the Velocity at D, and the Time of Descent thro' AD, will each be defined by $\sqrt{2AD}$.

3°. That, when $n=-2$, and $\dot{T}=\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, the Velocity $\left(\frac{\sqrt{ax-xx}}{x\sqrt{\frac{1}{2}a}}\right)$ will be as $\frac{DS}{CD\sqrt{\frac{1}{2}AC}}$, and the Time of Descent thro' AD, as $\sqrt{\frac{1}{2}AC}\times\overline{AS+DS}$.

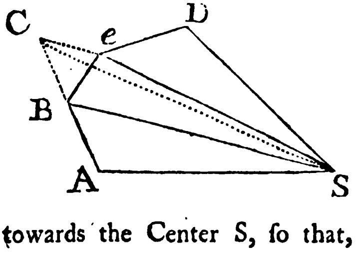
4°. And that, when $n = -3$, and $\dot{T} = \frac{ax\dot{x}}{\sqrt{a^2 - x^2}}$, the Velocity will be as $\frac{DR}{AC \times CD}$, and the Time as $AC \times DR$.

Hence the Time of the whole Descent thro' the Radius AC, appears to be as $\frac{AF}{AC}$, $\sqrt{2AC}$, $\sqrt{\frac{1}{2}AC} \times AF$, or AC^2. But the Time of one whole Revolution in the Periphery ARF &c, was found to be as $\frac{4AF}{AC^{\frac{n+1}{2}}}$ *; which in the four Cases above specified is $\frac{4AF}{AC}$, $\frac{4AF}{\sqrt{AC}}$, $4AF \times \sqrt{AC}$, and $4AF \times AC$: Therefore, if the Time of moving over the Quadrant AF be denoted by Q, it follows that the Time of Descent thro' the Radius AC, will be truly defined by Q. $Q \times \frac{AC\sqrt{2}}{AF}$, $Q \times \sqrt{\frac{1}{2}}$, or $Q \times \frac{AC}{AF}$ according to the foresaid Cases respectively.

* Art. 215.

LEMMA.

224. *The Areas which a revolving Body describes, by Rays drawn to the Center of Force, are proportional to the Times of their Description.*



For, let a Body, in any given Time, describe the Right-line AB, with an uninterrupted uniform Motion; but upon its Arrival at B let it be impelled towards the Center S, so that, instead of proceeding along

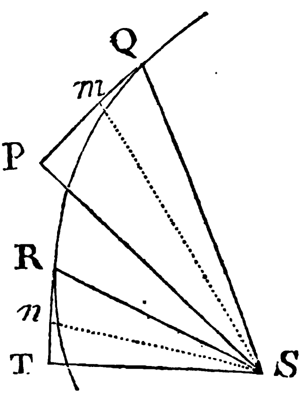
along ABC, it may, after the Impulse, describe the Right-line B*e*.

Becaufe the Force, acting in the Line SB, can neither add to, nor take from, the Celerity which the Body has in a Direction perpendicular to that Line, the Diftance of the Body from the faid Line, at the end of a given Time, will therefore be the very fame as if no Force had acted; and confequently the Area B*e*S equal to the Area BCS, which would have been defcribed in the fame time, had the Body proceeded uniformly along BC; becaufe Triangles, having the fame Bafe and Altitude, are equal.

Therefore feeing no Impulfe, however great, can affect the Quantity of the Area defcribed about the Center S, in a given Time, and becaufe the Areas ASB, BSC, defcribed about that Point, when no Force acts, are as the Bafes AB, BC, or the Times of their Defcription, the Propofition is manifeft.

Corollary.

225. Hence the Velocity of a revolving Body, at any Point Q or R, is inverfely as the Perpendicular SP or ST, falling from the Center of Force upon the Tangent at that Point.

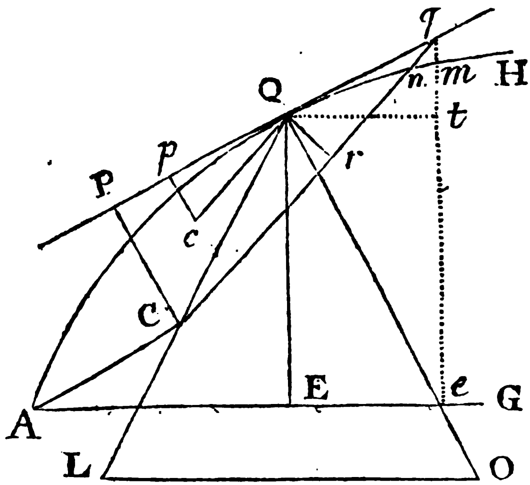


For, let two other Bodies *m* and *n* be fuppofed to move uniformly from Q and R, along the Tangents QP and RT, with Velocities refpectively equal to thofe of the revolving Body at Q and R; then the Diftances Q*m* and R*n*, gone over in the fame Time, will be to each other as thofe Velocities; and the Areas QS*m* and RS*n* will be equal, being equal to

to those described by the revolving Body in the same
* Art. 213. time * : Whence $Qm \times SP$ being $= Rn \times ST$, it follows
that $Qm : Rn :: ST : SP :: \frac{1}{SP} : \frac{1}{ST}$.

PROPOSITION IX.

226. *To determine the Law of the centripetal Force, tending to a given Point* C, *whereby a Body may describe a given Curve* AQH.



Let QP be a Tangent to the Curve at any Point Q; upon which, from the Center C, let fall the Perpendicular CP; put $CQ = s$, $CP = u$; and let the Velocity of the Projectile at Q be denoted by v.

Therefore, since v^2 is always as $\frac{1}{u^2}$ (by the *Corol.* to *Lemma*) it is evident, by taking the Fluxions of both Quantities, that $v\dot{v}$ will also be as $\frac{-\dot{u}}{u^3}$: But the centripetal Force, whether the Body moves in a Right-line or or a Curve, is always as $-\frac{v\dot{v}}{\dot{s}}$ (by *Art.* 219. and 206.)

Therefore the centripetal Force is likewise as $\frac{\dot{u}}{u^3\dot{s}}$. *Q.E.I.*

The same otherwise.

227. Let the Ray of Curvature QO be denoted by *R*: Then, because the centripetal Forces in Circles are known to be as the Squares of the Velocities directly and the Radii inversely *, it follows that the Force, tending to the Point Q, whereby the Body might be retained in its Orbit at Q, or in the Circle whose Radius is QO, will be defined by $\frac{1}{u^2} \times \frac{1}{R}$: Whence (by the Resolution of Forces) it will be CP (u) : CQ (s) :: $\frac{1}{u^2 R}$ (the Force in the Direction QO) : $\frac{s}{u^3 R}$, the Force in the Direction QC: Which, because $R = \frac{s\dot{s}}{\dot{u}}$ * will also be expressed by $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

* Art. 212.

* Art. 73.

Another Way.

228. Let *nq* be the indefinitely small Part of the Right-line C*q*, intercepted by the Curve and the Tangent Q*q*, expressing the Effect of the centripetal Force in the Time of describing the Area QC*n*. Now these Effects, or the Distances descended by means of Forces uniformly continued, are known to be in the duplicate Ratio of the Times *, or of the Areas denoting those Times †: Therefore, the centripetal Force at Q, or the Distance descended by means thereof in a given Time, will be as *nq* apply'd to the second Power of the Area QC*q*, or as $\frac{nq}{CP^2 \times Qq^2}$. This Expression is the same

* Art. 201.

† Art. 224.

with

with that given by Sir *Isaac Newton*, in his *Principia*, *Book* 1. *Prop.* 6. But, to adapt it to a fluxional *Calculus*; let QE be an Ordinate to the principal Axis AG; and let (as usual) AE $= x$, EQ$=y$, AQ$=z$, Ee (or Qt) $= \dot{x}$, Q$q = \dot{z}$; supposing eq (parallel to EQ) to intersect the Curve and the Tangent in m and q.

Since Qq is conceived indefinitely small (or in its nascent State) the Triangle nmq may be taken as rectilineal *; also the Angle n=CQP and the Angle m= Qqt: Whence, it will be (by Trigonometry) as *S.* CQP (n) : *S.* Qqt (m) :: $mq : nq$; that is, as $\frac{CP}{CQ} : \frac{Qt}{Qq}$ :: $mq : nq = \frac{CQ \times Qt \times mq}{CP \times Qq}$: Which substituted above gives $\frac{CQ \times Qt \times mq}{CP^3 \times \overline{Qq}^3}$ for the Measure of the centripetal Force at Q: But mq (supposing x to flow uniformly) is known to be as $-\ddot{y}$: Therefore the Force at Q, is as $\frac{CQ \times Qt \times -\ddot{y}}{CP^3 \times Qq^3}$, or its Equal $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$; where the Divisor ($u^3\dot{z}^3$) is as the Cube of (QC$q$) the Fluxion of the Area AQC.

* Art. 136.

The very same Theorem may likewise be deduced from that given by our second Method: For, since (R) the Ray of Curvature at Q is universally * $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, the Value of $\frac{s}{u^3R}$ (there found) will here, by Substitution, become $= \frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$.

* Art. 68.

This Expression, tho' in appearance less simple than $\frac{\dot{u}}{u^3 s}$, first found, is, for the general part, more commodious in Practice.

Co-

Corollary I.

229. If the Point C be ſo remote that all Right-lines drawn from thence to the Curve may be conſidered as parallel to each other, the Force will then (making Qr perpendicular to Cq) be as $\frac{-s\dot{x}\ddot{y}}{CQ\times\overline{Qr}|^3}$, or barely as $\frac{-\dot{x}\ddot{y}}{\overline{Qr}|^3}$; ſince s (CQ) in this Caſe may be rejected.

From this Expreſſion, which is general, in all Caſes where the Force acts in the Direction of parallel Lines, it appears that the Force, which always acting in the Direction of the Ordinate QE, would retain the Body in its Orbit, is every where as $\frac{-\ddot{y}}{\dot{x}^2}$; becauſe QC here coincides with QE, and Qr becomes $= \dot{x}$.

Corollary II.

230. Becauſe the Force, tending to the Point C, is univerſally as $\frac{CQ}{CP^3\times QO}$ (or $\frac{s}{u^3 R}$) the Force to any other Point c, will, by the ſame Argument, be as $\frac{cQ}{\overline{cp}|^3\times QO}$. Hence the Forces, to different Centers C and c (about which equal Areas are deſcribed in the ſame time) are to each other in the Ratio of $\frac{CP^3}{CQ}$ to $\frac{\overline{cp}|^3}{cQ}$ inverſely.

Corollary III.

231. Moreover, the Ratio of the Velocity at Q to the Velocity whereby the Body might revolve in a Circle about the Center C, at the Diſtance CQ, is eaſily deduced from hence: For, ſince the Celerity at Q is that whereby

whereby the Body might revolve in a Circle about the Center O, and the Forces tending to the Centers O and C are to each other as u (CP) and s (CQ); it therefore follows, if the Ratio sought be assumed as v to w, that $\frac{v^2}{QO} : \frac{w^2}{QC} :: u : s$ (*by Art.* 212.) Whence also $v^2 : w^2 :: u \times QO\ (uR) : s \times QC\ (s^2)$ and consequently $v : w :: \sqrt{\frac{uR}{ss}} : 1 :: \sqrt{\frac{u\dot{s}}{s\dot{u}}} : 1 :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{\frac{\dot{u}}{u}}$ (because $R = \frac{s\dot{s}}{\dot{u}}$).

The same Proportion may also be derived from *Corol.* 2. *Prop.* 7. For it is there proved that $v : w :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{-\frac{\dot{v}}{v}}$; and it appears from above, that $-\frac{\dot{v}}{v} = \frac{\dot{u}}{u}$: Whence the whole is manifest.

If OL be made perpendicular to QC, QL will be $\left(= \frac{CP \times QO}{CQ}\right) = \frac{uR}{s}$, and $\frac{QL}{CQ} = \frac{uR}{s^2}$; and therefore $v : w :: QL^{\frac{1}{2}} : CQ^{\frac{1}{2}}$: Which is another Proportion of the proposed Celerities.

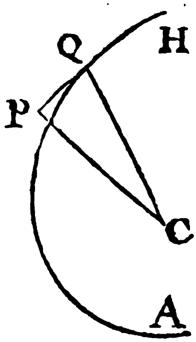
COROLLARY IV.

232. Lastly, the Law of centripetal Force being given, the Nature of the Trajectory AQ may from hence be found; for since the Force (F) is universally defined by $\frac{\dot{u}}{u^3\dot{s}}$, it is evident that $\frac{-1}{2u^2}$ will be = the Fluent of $F\dot{s}$; which, when F is given in Terms of s, will become known; and then, the Relation between u and s being given, the Curve itself is known.

EXAMPLE I.

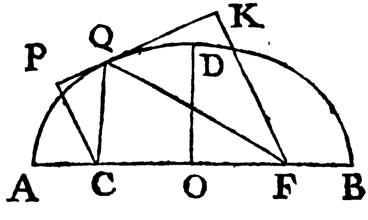
233. *Let the given Curve* AQH *be the logarithmic Spiral, and* C *the Center thereof:* Then u (CP) being in this Case $= \frac{bs}{a}$ *, we have $\frac{\dot{u}}{u^3\dot{s}}$ † $\left(= \frac{b\dot{s}}{a\dot{s}} \times \frac{a^3}{b^3s^3}\right)$ $= \frac{a^2}{b^2s^3}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ ‡ $\left(= \sqrt{\frac{bs\dot{s}}{a} \times \frac{a}{bs\dot{s}}}\right)$=Unity. Hence, it appears that the Force is inversely as the Cube of the Distance; and the Velocity, every where, equal to that whereby the Body might revolve in a Circle at the same Distance.

* Art. 61.
† Art. 227.
‡ Art. 232.



EXAMPLE II.

234. *Let it be required to find the Law of the centripetal Force, whereby a Body, tending to the Focus* C, *is made to revolve in the Periphery of an Ellipsis* AQDB.



From the other Focus F draw FK parallel to CP meeting the Tangent PQ (at Right-angles) in K, join F,Q; putting the transverse Axis AB $= a$, the Semi-conjugate OD $= \frac{1}{2} b$, and the Parameter $\left(\frac{b^2}{a}\right)$ $= p$: Then, CQ and CP being denoted as above *, we have FQ (=AB—CQ) $= a - s$; whence, by reason of the similar Triangles CQP and FQK, it will be

* Art. 231.

s:

$s : u :: a - s : FK = \frac{\overline{a-s} \times u}{s}$. But $FK \times CP$ is $= OD^2$ (by the Nature of the Curve.) Hence we get $\frac{\overline{a-s} \times u^2}{s} = \frac{1}{4} b^2$; and consequently $\frac{1}{u^2} = \frac{4a}{b^2 s} - \frac{4}{b^2}$; whereof the Fluxion being $- \frac{2\dot{u}}{u^3} = - \frac{4a\dot{s}}{b^2 s^2}$, we obtain $\frac{\dot{u}}{u^3 \dot{s}}$ * $= \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ † $= \sqrt{\frac{2 \times \overline{a-s}}{a}} = \sqrt{\frac{FQ}{AO}}$. Hence, it appears that the centripetal Force is, in this Case, as the Square of the Distance inversely; and the Velocity at Q is to that whereby the Body might describe a Circle at the Distance CQ, every where, in the Ratio of $FQ^{\frac{1}{2}}$ to $AO^{\frac{1}{2}}$.

* Art. 227.
† Art. 231.

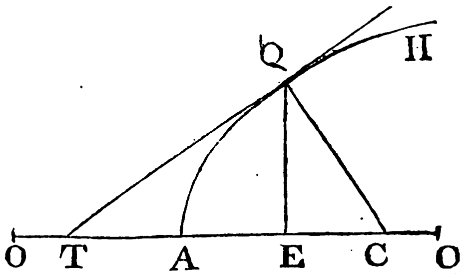
If the Curve had been an Hyperbola; then $\frac{a+s}{s} \times u^2$ (instead of $\frac{a-s}{s} \times u^2$) would have been $= \frac{1}{4} b^2$; and so $\frac{\dot{u}}{u^3 \dot{s}} = \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, the very same as before.

But, had it been a Parabola, the Equation would have been $\frac{a+0}{s} \times u^2 = \frac{1}{4} b^2$, or $\frac{u^2}{s}$ $\left(= \frac{b^2}{4a}\right) = \frac{1}{4} p$; and the Force, *still*, as $\frac{2}{ps^2}$. But, the Measure of the Velocity $\left(\sqrt{\frac{u\dot{s}}{s\dot{u}}} = \sqrt{\frac{2a - 2s}{a}}\right)$ in this Case becoming barely $= \sqrt{2}$, it follows that the Velocity in a Parabola is to that whereby the Body might describe a Circle at the same Distance from the Center, in the constant Ratio of $\sqrt{2}$ to Unity.

4 EX-

EXAMPLE III.

235. *Let it be required to find the Law of the centripetal Force, by which a Body, tending to any given Point* C, *in the Axis, is made to describe a conic Section* AQH.



Put the semi-transverse Axis (OA) $=a$, the semi-conjugate $=b$, and the given Distance of the Point C from the Vertex A $=c$: Put also the Abscissa AE, $=x$, the Ordinate EQ$=y$, and CQ$=s$ *(as before.)*

The Area of the Triangle ECQ being ($=\frac{1}{2}$EC×EQ) $= \frac{cy - xy}{2}$, its Fluxion is therefore $= \frac{c\dot{y} - x\dot{y} - y\dot{x}}{2}$; which added to $y\dot{x}$, the Fluxion of the Area AEQ, gives $\frac{c\dot{y} + y\dot{x} - x\dot{y}}{2}$ for the Fluxion of the whole Area ACQ described about the Center of Force. Whence (*by Art.* 228.) the required centripetal Force at Q will be as $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3}$. Which Expression is general, let the Curve be of what Kind it will. But in the Case above, y being $= \frac{b}{a}\sqrt{2ax \pm x^2}$, we have $\dot{y} = \frac{b\dot{x} \times \overline{a \pm x}}{a\sqrt{2ax \pm x^2}}$, $\ddot{y} = \frac{-ab\dot{x}^2}{\overline{2ax \pm x^2}|^{\frac{3}{2}}}$, and $c\dot{y} + y\dot{x} - x\dot{y} =$

$\frac{b\dot{x}\times\overline{ca+ax\pm cx}}{a\sqrt{2ax\pm x^2}}$; and therefore, by substituting these Values, we get $\frac{-s\dot{x}\dot{y}}{\overline{c\dot{y}+y\dot{x}-x\dot{y}}|^3}=\frac{a^4s}{b^2\times\overline{ca+ax\pm cx}|^3}$. Which, because $\frac{a^4}{b^2}$ is constant, will also be as $\frac{s}{\overline{ca+ax\pm cx}|^3}$. From whence it follows,

1°. If c be $=\mp a$, or the Center of Force be in the Center of the Section, the Force itself will be barely as ($\mp s$) the Distance.

2°. If it be in the Focus, then $ac+ax\pm cx$ becoming $=CQ\times a$, the Force will be inversely as the Square of the Distance.

3°. If the given Point be in the Vertex A, the Force will be as $\frac{s}{x^3}$: Which therefore in the Circle (where $x=\frac{s^2}{2a}$) will be as $\frac{1}{s^5}$, or the fifth Power of the Distance repricocally.

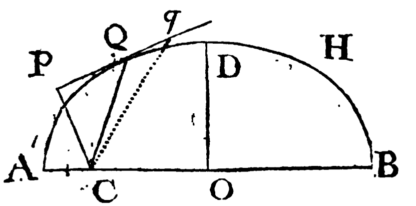
4°. Lastly, if the Point C be at an indefinite Distance from the Vertex, or the Force be supposed to act in the Direction of Lines parallel to the Axis AO; then the Force will be as the Cube of OE inversely.

PROPOSITION X.

236. *To determine the Ratio of the Velocities of Bodies revolving in different Orbits, about the same, or different, Centers; the Orbits themselves, and the Forces whereby they are described, being given.*

Let AQH be any Orbit, described about the Center of Force C, and let the Force itself at the principal Vertex A be denoted by F; also let r stand for the Semiparameter, or the Ray of Curvature at the Vertex, and let

let CP be perpendicular to the Tangent QP.



Then, the Celerity at A being, always, as $\sqrt{rF}$ (*by Art.* 212.) we have CP : CA :: $\sqrt{rF}$ (the Velocity at A) to $\frac{CA \times \sqrt{rF}}{CP}$, the Velocity at Q (*by Art.* 225.) Which answers in all Cases, let the Values of AC, r and F be what they will. *Q. E. I.*

Corollary I.

237. If the centripetal Force be as the Square of the Distance inversely, or F be expounded by $\frac{1}{AC^2}$, the Velocity at Q will become $\frac{AC}{CP} \times \sqrt{\frac{r}{AC^2}}$, or $\frac{\sqrt{r}}{CP}$: Whence the Velocities, in different Orbits, about the same Center, are in the subduplicate Ratio of the Parameters, and the inverse Ratio of the Perpendiculars from the Center of Force to the Tangents, conjunctly.

Corollary II.

238. Hence, if the Celerity at Q be denoted by Qq, and Cq be drawn; then, Qq being as $\frac{\sqrt{r}}{CP}$, it follows that $\sqrt{r}$ is as CP$\times$Qq, or as the Triangle QCq: There-

fore the Areas described about a common Center of Force in a given Time, are in the subduplicate Ratio of the Parameters.

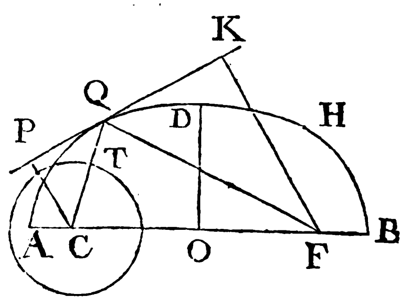
COROLLARY III.

239. Lastly, since the Area of the Curve AQHB &c. when an Ellipse *, is known to be as (AO×OD) AO× $\sqrt{r\times AO}$ (supposing O to be the Center) if the same be apply'd to $\sqrt{r}$, expressing the Area described in a given Part of Time (by the last *Corol.*) we shall thence have AO× $\sqrt{AO}$, or $AO^{\frac{3}{2}}$ for the Measure of the Time of one whole Revolution. From whence it appears, that the periodic Times, let the Species of the Ellipses be what they will, are in the sesquiplicate Ratio of their principal Axes.

* Art. 234.

PROPOSITION XI.

240. *The centripetal Force, tending to a given Point* C, *being as the Square of the Distances reciprocally, and the Direction and Velocity of a Body at any Point* Q *being given; to determine the Path in which the Body moves, and the periodic Time, in case it returns.*



It is evident from *Art.* 234. *and* 235. that the Trajectory AQB is a conic Section; whereof the Point C is one of the *Foci*.

Let

Let F be the other Focus, and upon the Tangent PQK let fall the Perpendiculars CP and FK, and let CQ and FQ be drawn: Alſo put the ſemi-tranſverſe Axis AO $= a$, the given focal Diſtance CQ $= d$, and the Sine of the Angle of Direction CQP (to the Radius 1) $= m$; and let the given Velocity at Q be to that whereby the Body might revolve in a Circle about the Center C, at that Diſtance, in any given Ratio of n to Unity: Then it will be $n : 1 :: FQ^{\frac{1}{2}} : AO^{\frac{1}{2}}$ (by *Art.* 234.) therefore $n^2 : 1^2 : FQ\ (2a-d) : AO\ (a)$; whence AO ($a$) is given $= \frac{d}{2-n^2}$. Moreover, ſince $CP = m \times CQ$, and $FK = m \times FQ$, we have OD^2 ($= CP \times FK = m^2 \times CQ \times FQ = \frac{m^2 n^2 d^2}{2-n^2}$; whence the ſemi-conjugate Axis (OD) is given likewiſe.

Laſtly, it will be (by *Art.* 239.) as $CT^{\frac{3}{2}} : AO^{\frac{3}{2}} ::$ (P) the periodic Time in any given Circle, whoſe Radius is CT, to $\left(\frac{AO^{\frac{3}{2}}}{CT^{\frac{3}{2}}} \times P\right)$ the required Time of one Revolution when the Orbit is an Ellipſis; that is, when n^2 is leſs than 2: For, if n^2 be $= 2$, the Curve (as its Axis $\frac{2d}{2-n^2}$ becomes infinite) will degenerate to a Parabola; and, if n^2 be greater than 2, the Axis being negative, it is then an Hyperbola; whoſe two principal Diameters are equal to $\frac{2d}{n^2-2}$ and $\frac{2mnd}{\sqrt{n^2-2}}$. *Q. E. I.*

COROLLARY.

241. Seeing neither the Value of AO, nor that of the periodic Time, is affected with m, it follows that the principal Axis, and the periodic Time, will remain

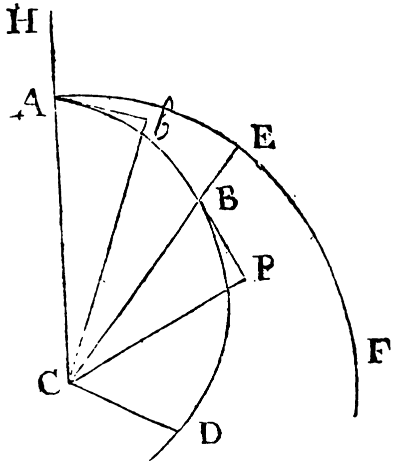
invariable, if the Velocity at Q be the same, let the Direction at that Point be what it will.

The same Solution may likewise be brought out, from *Art*. 238. by first finding the *principal Parameter*: For, it is evident that the Area described by the Body about the Center C, in any given Time, is to the Area described, in the same Time, by another Body revolving in a Circle at the Distance CQ, as mn to Unity: Hence, it will be $1^2 : m^2n^2 :: d : (m^2n^2d)$ the Semi-parameter *: From which (proceeding as above) we get $a \times m^2n^2d$ $(= OD^2) = m^2 \times \overline{2ad - d^2}$; and consequently $a = \frac{d}{2 - n^2}$, the same as before.

* Art. 238.

PROPOSITION XII.

242. *The centripetal Force being as any Power (n) of the Distance, and the Direction and Velocity of a Body at any Point* A *being given, to determine the Orbit or Trajectory.*



From the Center of Force C, to any Point B in the required Trajectory ABD, let CB be drawn; join C, A, and let A*b* be the given Direction of the Body at the Point A, and C*b* perpendicular thereto; also let the Velocity at A be to that whereby a Body might describe a Circle AEF, about the Center C, in any given Ratio of p to Unity; putting CA$=a$, and CB$=x$: Then, be-

becauſe this laſt Velocity (the centripetal Force being as x^n (or a^n) is rightly defined by $a^{\frac{n+1}{2}}$ *, the Velocity of the Body at A will be truly expreſſed by $pa^{\frac{n+1}{2}}$. * Art. 214.

Moreover, it is proved in *Art.* 221. *and* 206. that if the Celerity, at any given Diſtance a from the Center, be denoted by c, the Celerity at any other Diſtance x will be truly repreſented by $\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$: Whence, $pa^{\frac{n+1}{2}}$ being ſubſtituted for c, we have $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}$ for the Celerity at B.

But now, to determine the Curve itſelf from hence, let BP be a Tangent to it at B, and CP perpendicular to BP; alſo let CB, produced, meet the Periphery of the Circle in E; putting the Arch AE$=z$, the foreſaid Velocity at B (to ſhorten the Operation) $=v$, and Cb=b: Then it will be (by *Art.* 225.) $v : c$ (the Velocity at A) :: b : CP $= \frac{bc}{v}$: Whence BP ($= \sqrt{CB^2 - CP^2}$) $= \frac{\sqrt{x^2v^2 - b^2c^2}}{v}$.

Moreover (by *Art.* 35.) we have, as CB : CP :: v : ($\frac{CP}{CB} \times v$) the Velocity of the Body at B in a Direction perpendicular to CE; and conſequently, as CB : CE :: $\frac{CP}{CB} \times v$ (the ſaid Velocity) to $\frac{CP \times CE}{CB^2} \times v$ the angular Velocity of the Point E (revolving with the Body.) By the ſame *Article*, the Velocity at B in the

Direction CBE will be $\frac{BP}{CB} \times v$: Therefore, the Velocity of E being to the Velocity of B, in the said Direction, as $\frac{CP \times CE}{CB^2}$ to $\frac{BP}{CB}$, the Fluxions of AE (z) and CB (x) must consequently be in that Ratio; that is, $\frac{CP \times CE}{CB^2} : \frac{BP}{CB} :: \dot{z} : \dot{x}$; whence $\dot{z} = \frac{CP \times CE}{CB \times BP} \times \dot{x} = \frac{bc}{v} \times \frac{a}{x} \times \frac{v\dot{x}}{\sqrt{x^2v^2 - b^2c^2}} = \frac{abc\dot{x}}{x\sqrt{x^2v^2 - b^2c^2}} = \frac{ab\dot{x}}{x\sqrt{\frac{x^2v^2}{c^2} - b^2}}$. Which Equation is general, let the Law of the centripetal Force be what it will: But in the Case above proposed, v^2 being $= \overline{p^2 + \frac{2}{n+1} \times a^{n+1}} - \frac{2x^{n+1}}{n+1}$, and $c^2 = p^2a^{n+1}$; it becomes $\dot{z} = \frac{abp\dot{x}}{x\sqrt{\overline{p^2 + \frac{2}{n+1}} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{\overline{n+1} \times a^{n+1}}}}$; whose Fluent is the Measure of the angular Motion; from which, when found, the Orbit may be constructed: Because, when AE, or the Angle ACE is given, as well as CB, the Position of the Point B is also given. But this Value of $\dot{z}$ is indeed too complex to admit of a Fluent in algebraic Terms, or even by circular Arcs and Logarithms, except in certain particular Cases; as when the Exponent n is equal to 1, -2, -3, or -5; besides some others wherein the Values of p and n are related in a particular Manner. *Q. E. I.*

Co-

COROLLARY I.

243. If the given Velocity at A be ſuch that $p^2 + \frac{2}{n+1} = 0$, or $p = \sqrt{\frac{-2}{n+1}}$ (which is always poſſible when the Value of $n+1$ is negative) our Equation will become $\dot{z} = \frac{abp\dot{x}}{x\sqrt{-p^2b^2 + \frac{p^2x^{n+3}}{a^{n+1}}}}$: Which, by putting $n+3 = m$, &c. is reduced to $\dot{z} = \frac{ab\dot{x}}{x\sqrt{-b^2 + \frac{x^m}{a^{m-2}}}}$.

Whereof the Fluent will be found (by the ſecond Part of this Work) equal to $\pm\frac{2a}{m}$ multiply'd by the Difference of the two circular Arcs, whoſe Secants are $\frac{x^{\frac{1}{2}m}}{ba^{\frac{1}{2}m-1}}$ and $\frac{a}{b}$, to the Radius Unity. From this Value of the Arch AE the Poſition of the Point B, in the Orbit, is given.

But if the Angle of Direction CAb be a right one, the Fluent will become barely $= \pm\frac{2a}{m} \times$ Arch whoſe Secant is $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}m}}$ (becauſe then $b = a$, and the Arch whoſe Secant is $\frac{a}{b}$, $= 0$) which therefore when $x^{\frac{m}{2}}$ becomes in-

infinite, will be truly defined by $\pm \frac{1}{2m} \times$ whole Periphery AF, *&c.* Whence it is evident that the Body must either fly intirely off, or fall to the Center C, in a Number of Revolutions expressed by $\pm \frac{1}{2m}$; according as the Value of m is positive or negative.

Thus, if $n = -2$, and $m = 1$, the Body will fly intirely off in half a Revolution: And, if $n = -4$, and $m = -1$, it will fall to the Center in half a Revolution.

COROLLARY II.

244. Moreover, tho' the Fluent expressing the Angle at the Center cannot be exhibited in a general Manner, yet there are certain Cases of the Exponent (n) where its respective Values may be derived from each other.

For let (as above) $n + 3$ be put $= m$, and (to shorten the Operation) let CA (a) be taken as Unity: Then our Equation will be transformed to $\dot{z} = \frac{b\dot{x}}{x\sqrt{1 + \frac{2}{m-2.p^2} \times x^2 - b^2 - \frac{2x^m}{m-2.p^2}}}$: Make $y = x^{\frac{m}{2}}$, and it will be farther transformed to $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1 + \frac{2}{m-2.p^2} \times y^{\frac{4}{m}} - b^2 - \frac{2y^2}{m-2.p^2}}}$:

Put $r = \frac{4}{m}$, and it will become $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{\frac{ry^2}{r-2.p^2} - b^2 + 1 - \frac{r}{r-2.p^2} \times y^r}}$: Lastly,

let

let $\frac{r}{r-2.p^2} = 1 + \frac{2}{r-2.q^2}$ (or $1 - \frac{r}{r-2.p^2} = -\frac{2}{r-2.q^2}$, or $q^2 = \frac{2p^2}{r-p^2 \times r-2}$) and then we shall have $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1+\frac{2}{r-2.q^2} \times y^2 - b^2 - \frac{2y^r}{r-2.q^2}}}$.

Which Expression (excepting the general Multiplicator $\frac{2}{m}$) being exactly of the same Form with the first above given, must therefore be the Fluxion of the Angle at the Center, when the Index of the Force is $r-3$; for the very same Reasons that the former appears to be the Fluxion thereof when the Index is $m-3$ (or n.)

Hence, if the Fluent of

$$\frac{b\dot{y}}{y\sqrt{1+\frac{2}{r-2.q^2} \times y^2 - b^2 - \frac{2y^r}{r-2.q^2}}}\text{, or the}$$

Angle at the Center, when the Exponent is $r-3$ (or $\frac{4}{m} - 3 = \frac{4}{n+3} - 3$) be denoted by w, the Value of z, (the Measure of the said Angle, when the Exponent is $m-3$ (or n) will be truly defined by $\frac{2w}{m}$.

From which we collect that, if the Indices of the Force, in any two Cases, be represented by n and $\frac{4}{n+3} - 3$, and the respective Distances from the Center by x and $x^{\frac{n+3}{2}}$, then the Angles themselves corresponding to those Distances will be every where in the constant Ratio of 2 to $n+3$. Therefore, when the Orbit can be be constructed in the one Case, it also may in the other, provided the above Equation $q^2 \left(= \frac{2p^2}{r - p^2 \times \overline{r-2}}\right) = \frac{\overline{n+3}\cdot p^2}{2 + \overline{n+1}\cdot p^2}$, for the Relation of the Celerities at A, does not become impossible, as it will, sometimes, when n is a negative Number.

COROLLARY III.

245. If the Body be supposed to move in a vertical Direction AH; then, putting the Velocity $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}} = 0$, we get x (CH) $= \overline{\tfrac{1}{2}p^2 \times \overline{n+1} + 1}\big|^{\frac{1}{n+1}} \times a =$ the Height to which the Body will ascend: Hence $\overline{\tfrac{1}{2}p^2 \times \overline{n+1} + 1}\big|^{\frac{1}{n+1}} \times a - a$ (=AH) is the Distance thro' which it must freely descend to acquire the given Celerity at A: This Distance, in case of an uniform Force, when $n = 0$, will become $= \frac{1}{2}p^2 a$: And, when the Force is inversely as the Square of the Distance, it will then be $= \frac{p^2 a}{2 - p^2}$.

But, when $p = 1$, or the Velocity at A is just sufficient to retain a Body in the Circle AEF, AH becomes $= \overline{\frac{3+n}{2}}\big|^{\frac{1}{n+1}} \times a - a$: Which in the two Cases aforesaid will be equal to $\frac{1}{2}a$, and a respectively; but, infinite, when n is $= -3$.

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COROLLARY IV.

246. When the Value of $n+1$ is positive, the Velocity at the Center, where $x=0$, will be barely $= \sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; but if the Value of $n+1$ be negative, the Velocity at the Center will be infinite; becauſe, then 0^{n+1} is infinite.

COROLLARY V.

247. Moreover, when $n+1$ is negative and x infinite, the Velocity alſo becomes $=\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; becauſe then $x^{n+1}=0$.

Hence, if the centripetal Force be inverſely as ſome Power of the Diſtance greater than the firſt, the Body may aſcend, *ad infinitum*, and have a Velocity always greater than $\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; which is to, $pa^{\frac{n+1}{2}}$, the given Velocity, at A, as $\sqrt{p^2+\frac{2}{n+1}}$ to p. And this will actually be the Caſe when the Value of $p^2+\frac{2}{n+1}$ is poſitive, or p^2 greater than $\frac{2}{-n-1}$, but not otherwiſe, the ſquare Root of a negative Quantity being impoſſible.

Thus, if $n=-2$, or the Force be inverſely as the Square of the Diſtance, and p^2, at the ſame time, greater than $2\left(\frac{2}{-n-1}\right)$ the Body will not only continue to aſcend *in infinitum*, but have a Velocity always greater than that defined by $\sqrt{p^2-2}$, which is its Limit.

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Corollary VI.

248. Hence the least Celerity sufficient to cause the Body to ascend for ever in a Right-line is given. For, putting $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}} = 0$, we have $p = \sqrt{\frac{2}{-n-1}}$. Therefore the least Celerity by which the Body might ascend for ever, is to that whereby it may revolve in a Circle AEF, as $\sqrt{\frac{2}{-n-1}}$ to Unity. From which it appears that, if the Force be inversely as any Power of the Distance greater than the third, a less Velocity will cause a Body to ascend *ad infinitum* than would retain it in a Circle.

Scholium.

249. From the Ratio of the Velocity $\left(\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}\right)$ wherewith the Body arrives at any Distance x from the Center, to that $\left(x^{\frac{n+1}{2}}\right)$* which it ought to have to revolve in a Circle at the same Distance, it will not be difficult to determine in what Cases the Body will be forced to the Center, and in what others it will continue to fly it *ad infinitum*.

* Art. 214.

For, first, if the Angle CAb be acute, or the Body from A begins to descend, it will continue to do so till it actually arrives at the Center, if the former Velocity, during the Descent, be not somewhere greater than the latter, or the Quotient $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ greater than Unity; because, if it ever begins to ascend, it it must have an *Apse*, as D (where a Right-line drawn from the Center cuts the Orbit at Right-angles) and there the Celerity must evidently be greater than that sufficient to cause the Body to revolve in a Circle.

Secondly, but if the Quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, in the Access of the Body towards the Center, increases so as to become greater than Unity, or be every where so; then the Velocity at all inferior Distances being more than sufficient to retain a Body in a Circle at any such Distance, the Projectile cannot be forced to the Center.

After the same Manner, if the Angle CA*b* be obtuse, or the Body from A begins to ascend, it will continue to do so for ever, when the foresaid Quantity is always greater than Unity, or, which is the same, when the Body, in its Recess from the Center, has in every Place thro' which it passeth, a Velocity greater than sufficient to retain it in a Circle at that Distance.

It therefore now remains to find in what Laws of the centripetal Force these different Cases obtain: And, first, it is easy to perceive that when the Value of $n+1$ is positive, that of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, by increasing x, become equal to nothing. Therefore the Body cannot ascend for ever in this Case: Neither can it descend to the Center (except in a Right-line) because the foresaid Quantity, by diminishing x, becomes greater than Unity (or any other assignable Magnitude.)

But, if the Value of n be betwixt -1, and -3, the said general Expression, taking x infinite, will also become infinite, provided the Value of $p^2 + \frac{2}{n+1}$ be positive (or p^2 greater than $\frac{2}{-n-1}$). Therefore the

Body

Body, in this Case, may ascend *ad infinitum*, but cannot possibly fall to the Center (except in a Right-line) since, $\sqrt{-\frac{2}{n+1}}$, the Value of the general Expression, when $x=0$, is greater than Unity.

Lastly, if n be expressed by any negative Number greater than -3, or the Law of the Force be inversely as any Power of the Distance greater than the third, the two extreme Values of $\sqrt{p^2+\frac{2}{n+1}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$ will, *still*, be denoted as in the preceding Case; but here the latter of them, $\sqrt{\frac{-2}{n+1}}$, is less than Unity. Therefore the Body must, in this Case, either ascend for ever, or be forced to the Center; except in one particular Circumstance, hereafter to be taken notice of.

Now, from these Observations we gather,

1°. That, when the centripetal Force is as any Power of the Distance directly, or less than the first Power thereof inversely, the Orbit will always have an higher and a lower *Apse*; beyond which the Body cannot ascend or descend.

2°. That, when the centripetal Force is inversely as any Power of the Distance (whole or broken) betwixt the first and third, the Orbit will also have two *Apsides*, if p be less than $\sqrt{-\frac{2}{n+1}}$; but otherwise, only one; in which last Case the Body, after it has passed its Apse, will continue to recede from the Center *in infinitum*.

3°. That when the Force is inversely as any Power greater than the third, the Orbit can, at most, have but one *Apse*; but, in some Cases, it will have none at all: And it may be worth while to inquire here, under what Restrictions of the Velocity (p) this will happen; since thereby, besides being able to know when the Body will be be forced to the Center, &c. we ſhall fall upon a Circumſtance ſomewhat remarkable and curious.

Now it appears, that, if the Body from A begins to deſcend, it muſt, when it comes to an *Apſe* at D, have a Velocity there greater than is ſufficient to retain it in a Circle; in which Caſe the general Expreſſion $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ (ſo often mention'd above) muſt accordingly be greater than Unity. Let it be therefore made equal to Unity, which is the utmoſt Limit thereof, beyond which the Orbit cannot admit of an *Apſe*; putting at the ſame time $\dot{x}$, or its Diviſor $\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{\overline{n+1}.a^{n+1}}}$, in the general Equation of the Orbit, equal to nothing (it being always ſo at the *Apſides.*) Then, from theſe two Equations, duly order'd, we ſhall get $x = \left(\frac{2+\overline{n+1}.p^2}{n+3}\right)^{\frac{1}{n+1}} \times a$, and p^2 $\left(= \frac{x^{n+3}}{a^{n+1}}\right) = \left(\frac{2+\overline{n+1}.p^2}{n+3}\right)^{\frac{n+3}{n+1}} \times \frac{a^2}{b^2}$. Now, it is evident, if the Value of p be greater than is given from the laſt Equation, the Orbit will have an Apſe; but if leſs, it can have none. In the former Caſe, the Body will therefore fly quite off; and in the latter, it will be forced to the Center. But we are now, naturally, led to inquire what will be the Conſequence when the Value of p is neither greater nor leſs, but exactly the ſame as given from the foreſaid Equation: This is the Caſe above hinted at; and here the Body will continue to deſcend for ever in a Spiral, yet never ſo low as to enter within the Circle whoſe Radius CD is $= \left(\frac{2+\overline{n+1}.p^2}{n+3}\right)^{\frac{1}{n+1}} \times a$. For, if the contrary were possible, the Body, at its Arrival to the Circumference of that Circle, would (because of the foresaid Equations) not only have a Direction, but also Velocity proper to retain it therein; which cannot be, because the Parts of the Orbit on either Side of an Apse are always similar to each other.

From the same Equation, the Value of the Limit will also be given when the Angle of Direction CAb is obtuse, or the Body is projected upwards:

For that Equation (as is easy to demonstrate *) admits of two different Roots, or Values of p; the one greater, the other less, than Unity: Whereof the former, giving CD (x) less than CA, is to be taken in the preceding Case, and the latter (making CD greater than CA) in the present. And the Body will, either, continue to ascend for ever, or come to an *Apse*, and from thence fall to the Center, according as the given Value of p is greater or less than that here specified. But if it be neither greater nor less, but exactly the same, then the Body, tho' it will still continue to ascend for ever in a Spiral, yet it can never rise so high as the Circumference of the Circle whose Radius CD is $=$

$\left(\frac{2+\overline{n+1}.p^2}{n+3}\right)^{\frac{1}{n+1}} \times a$, for Reasons similar to those already deliver'd, in respect to the preceding Case.

* *Mathematical Dissert.* p. 167.

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